Today:

- Probabilistically Checkable Proof Systems
- Proofs of NP statements can be verified with $O(1)$ queries!

Useful Fact:

Given vectors $\bar{a} \neq \bar{b}$

$$\Pr_{\bar{r} \in \{0,1\}}[\bar{a} \cdot \bar{r} \neq \bar{b} \cdot \bar{r}] \geq 1/2$$

Given matrices A, B, C

if $A \cdot B \neq C$ then

$$\Pr_{\bar{r}}[\underbrace{A \cdot \underbrace{B \cdot \bar{r}}}_{O(n^2) \text{ time}} \neq C \cdot \bar{r}] \geq 1/2$$

also true for equality mod 2

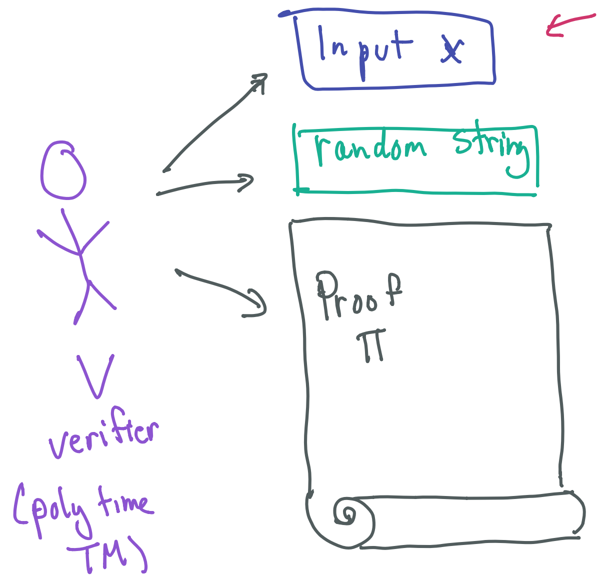
why?

Homework 1 optional problem 2

also: same argument used to show Fourier basis is orthogonal

$\langle \chi_S, \chi_T \rangle = 0$ for $S \neq T$

Probabilistically Checkable Proofs



← theorem to prove } for today x is 3-CNF satisfiable

<u>Thm</u> x is satisfiable

fixed fctn

differs from prover because can't adapt to V's questions

created by adversary with unlimited computational power

<u>def</u> $L \in PCP(r, q)$ if $\exists V$ s.t.

- poly time TM
- uses $\leq r(n)$ random bits
- uses $\leq q(n)$ queries to π ↑ 1 bit each

1) $\forall x \in L \quad \exists \pi$ s.t.

$$\Pr_{\text{V's random strings}}[\, V, \pi \text{ accepts} \,] = 1$$

2) $\forall x \notin L \quad \forall \pi' \quad \Pr_{\text{V's random strings}}[\, V, \pi' \text{ accepts} \,] < 1/4$

SAT $\in$ PCP(0, n) ← look at all settings of vars

<u>Today</u>:

<u>Thm</u> $NP \subseteq PCP(O(n^3), O(1))$

$O(n^3)$ ← \$; $O(1)$ ← queries

<u>Actually</u>: <u>Thm</u> $NP \subseteq PCP(O(\log n), O(1))$

3SAT: $F = \bigwedge C_i$ s.t. $C_i = (y_{i_1} \vee y_{i_2} \vee y_{i_3})$

where $y_{i_j} \in \{x_1 .. x_n\ \bar{x}_1 ... \bar{x}_n\}$

I. Encode satisfiability of F as a collection of polys in variables of assignment

- one for each clause
- low degree
- evaluate to 0 if assignment satisfies clause
- V knows coeffs – depend on structure of clause & vars of clause

Arithmetization of 3SAT:

boolean formula F $\Longleftrightarrow$ arithmetic formula $A(F)$ over $\mathbb{Z}_2$

$$T \longleftrightarrow 1$$
$$F \longleftrightarrow 0$$
$$x_i \longleftrightarrow x_i$$
$$\overline{x_i} \longleftrightarrow 1 - x_i$$
$$\alpha \wedge \beta \longleftrightarrow \alpha \cdot \beta$$
$$\overline{\bar{\alpha} \wedge \bar{\beta}} = \alpha \vee \beta \longleftrightarrow 1 - (1-\alpha)(1-\beta)$$
$$\alpha \vee \beta \vee \gamma \longleftrightarrow 1 - (1-\alpha)(1-\beta)(1-\gamma)$$
$$\overline{(\alpha \vee \beta \vee \gamma)} \longleftrightarrow (1-\alpha)(1-\beta)(1-\gamma)$$

example:

$$x_1 \vee \bar{x}_2 \vee x_3 \Leftrightarrow 1 - (1-x_1)(1-(1-x_2))(1-x_3)$$
$$= 1 - (1-x_1)(x_2)(1-x_3)$$

F satisfied by $\bar{a}$ iff $A(F)(\bar{a}) = 1$

Consider $\mathcal{C}(\bar{x}) = (\hat{C}_1(\bar{x}), \hat{C}_2(\bar{x}), \dots)$

Note: (1) Complements of arithmetization of clause C_i

$\Rightarrow$ evaluate to 0 if x satisfies C_i

(2) each $\hat{C}_i$ is deg ≤ 3 poly in x

(3) V knows coeffs of each $\hat{C}_i$

Need to convince V that

$$\mathcal{C}(\bar{a}) = (0, 0, \dots 0)$$

w/o sending $\bar{a}$

"weird idea"

assume $\exists$ "little birdie" who tells V dot products of $\mathcal{C}$ with random vectors mod 2

(V inputs $\bar{r}$
birdie answers $\mathcal{C}(\bar{a}) \cdot \bar{r}$)

Fix $\bar{a}$

$$\left(\hat{C}_1(\bar{a}), \ldots, \hat{C}_m(\bar{a})\right) \cdot (r_1 \ldots r_m)$$

$$\equiv \sum r_i \hat{C}_i(\bar{a}) \bmod 2$$

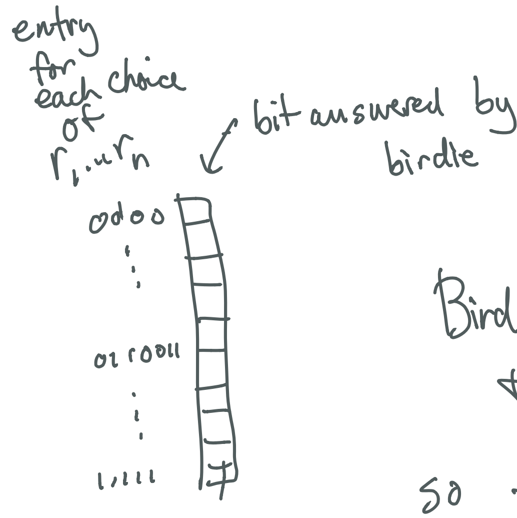
$$\Pr\left[\sum r_i \hat{C}_i(\bar{a}) = 0\right] = \begin{cases} 1 & \text{if } \forall_i \ \hat{C}_i(\bar{a}) = 0 \\ 1/2 & \text{o.w.} \end{cases}$$

$(\exists i$ s.t. $\hat{C}_i(\bar{a}) \neq 0$

$\Downarrow$

$C(\bar{a})$ not satisfied$)$

At this point can write a very long proof



Birdie can cheat & always answer 0!!

so far - no check for consistency with $\hat{C}_i(\bar{a})$

So, why believe the birdie?

recall:

we know r_i's

we know coeff of polys of $\hat{C}_i$'s

$\hat{C}_i$'s have deg ≤ 3 in a_i's

we <u>do</u> not know a_i's

V doesn't know these

$$\sum_i r_i \hat{C}_i(a) = \Gamma + \sum_i a_i \alpha_i + \sum_{i,j} a_i a_j \beta_{ij} + \sum_{ijk} a_i a_j a_k \gamma_{ijk} \pmod 2$$

- V knows these (s, does proof): depend on r_i's, coeffs of polys; do <u>not</u> depend on a_i's
- since working mod 2, all values $\in \{0,1\}$

from here on:

$$\left.\begin{aligned} \alpha_i &\to x_i \\ \beta_{ij} &\to y_{ij} \\ \gamma_{ijk} &\to z_{ijk} \end{aligned}\right\} \text{no reln to vars of 3SAT}$$

<u>idea</u>: make birdie write all answers for all choices of r_i's & check consistency (and later check satisfying the assignment)

We will do something <u>stronger</u> & <u>easier to check</u>

better idea

make birdie write out answers to all

3 seperate parts of proof

- linear fctns. of $\bar{a}$
- deg 2 " " "
- deg 3 " " "

- we only care about 1 lin / deg 2 / deg 3 fctn of $\bar{a}$

- will use to check that birdie wrote down a proper encoding of $\bar{a}$

V doesn't know

V knows

def $A : \mathbb{F}_2^n \to \mathbb{F}_2$ $\quad A(\bar{x}) = \sum a_i x_i = a^T \cdot \bar{x}$

$B : \mathbb{F}_2^{n^2} \to \mathbb{F}_2$ $\quad B(\bar{y}) = \sum_{i,j} a_i a_j y_{ij} = \underbrace{(a \circ a)}_{\text{outer product}}^T \cdot \bar{y}$

if $z = b \circ c$
$z_{ij} := b_i \cdot c_j$

$C : \mathbb{F}_2^{n^3} \to \mathbb{F}_3$ $\quad C(\bar{z}) = \sum_{ijk} a_i a_j a_k z_{ijk}$
$= (a \circ a \circ a)^T \cdot \bar{z}$

Proof contains:

Complete description of truth tables of $\tilde{A}, \hat{B}, \tilde{C}$ for all inputs $\bar{x}, \bar{y}, \bar{z}$

supposed to be A, B, C

but V needs to check

What to check?

(1) $\tilde{A}, \hat{B}, \tilde{C}$ are of right form

- all are linear fctns $\Rightarrow$ sc-$\tilde{A}$ will always answer according to closest lin fctn
  - linearity test + self-correct passes if $\tilde{A}$ close to linear
- correspond to same assignment $\bar{a}$
  - test all self-corrections consistent

(2) $\bar{a}$ is a sat assignment

all $\hat{C}_u$'s evaluate to 0 on $\bar{a}$

How to do (1):

- Test $\hat{A}, \hat{B}, \tilde{C}$ are all $\frac{1}{8}$ close to linear fctns
  - Pass if linear
  - Fail if $\geq \frac{1}{8}$-far from linear

  in $O(1)$ queries

#random bits: $O(n^3)$
#queries $O(1)$
runtime $O(n^3)$

- From now on use self-corrector to get $sc\text{-}\hat{A}$, $sc\text{-}\hat{B}$, $sc\text{-}\tilde{C}$ lin fctns

  can query on all inputs

  (use really small error bound on S-C s.t. if union bound over all calls to $Sc\hat{A}$ $Sc\hat{B}$ + $Sc\tilde{C}$ will never see error)

per query to self-corr:
#random bits $O(n^3)$
queries $O(1)$
runtime $O(n^3)$

Consistency Test:

are $sc\text{-}\tilde{A}$, $sc\text{-}\tilde{B}$ + $sc\text{-}\tilde{C}$ from <u>same</u> assignment $\bar{a}$?

Tester:

assume $\tilde{A}$ + $\tilde{B}$ + $\tilde{C}$ correspond to <u>same</u> $\bar{a}$

pick random $\bar{x}_1\ \bar{x}_2\ \ \bar{x}\ \bar{y}$

test that $sc\text{-}\tilde{A}(\bar{x}_1)\cdot sc\text{-}\tilde{A}(\bar{x}_2)$

$$= \sum a_i x_{1i} \cdot \sum a_j x_{2j}$$

$$= \sum_{ij} a_i a_j\, x_{1i} x_{2j}$$

$$= sc\text{-}\tilde{B}(\bar{x}_1 \circ \bar{x}_2)$$

random bits $O(n^2)$

queries $O(1)$

runtime $O(n^3)$

test that $sc\text{-}\tilde{A}(\bar{x})\cdot sc\text{-}\tilde{B}(\bar{y}) =$

$$= \sum_i a_i x_i \cdot \sum_{jk} a_j a_k y_{jk}$$

$$= \sum a_i a_j a_k\, x_i y_{jk}$$

$$= sc\text{-}\tilde{C}(\bar{x} \circ \bar{y})$$

<u>note</u>:

not unif dist queries

<u>but</u> s-c helps here

Is it a good test?

given sc-$\tilde{A}$, sc-$\hat{B}$, sc-$\tilde{C}$ $\Big\}$ all lin fctns

$A(x) = a^T x$

$B(y) = b^T y$

$C(y) = c^T z$

hopefully $b^T = (a \circ a)^T$

$c^T = (a \circ b)^T$

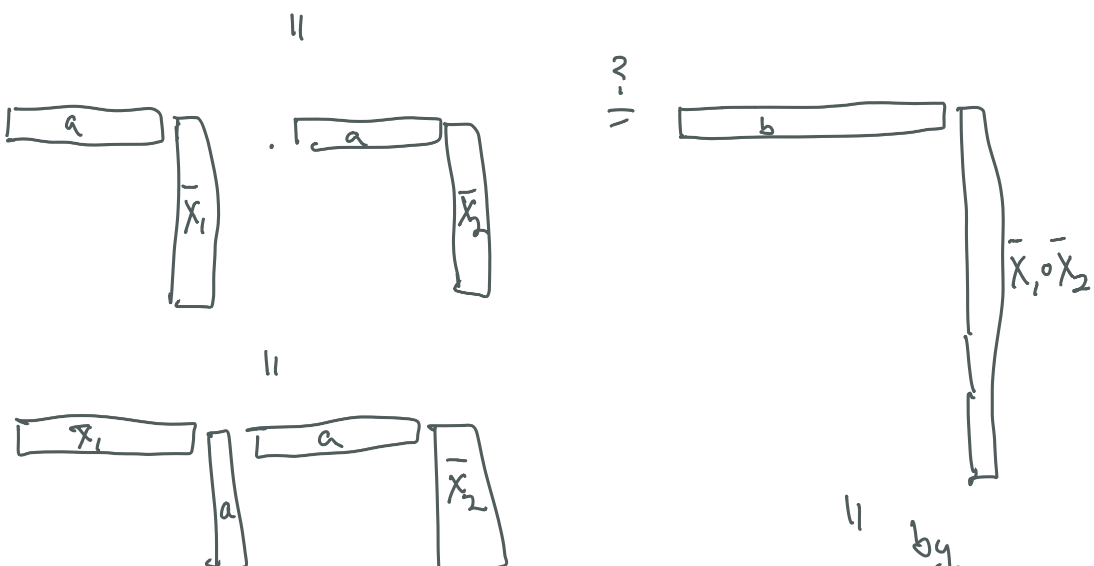
$\stackrel{?}{=} (a \circ a \circ a)^T$

If $b = a \circ a$, $c = a \circ a \circ a$ then test pass via green argument ✓

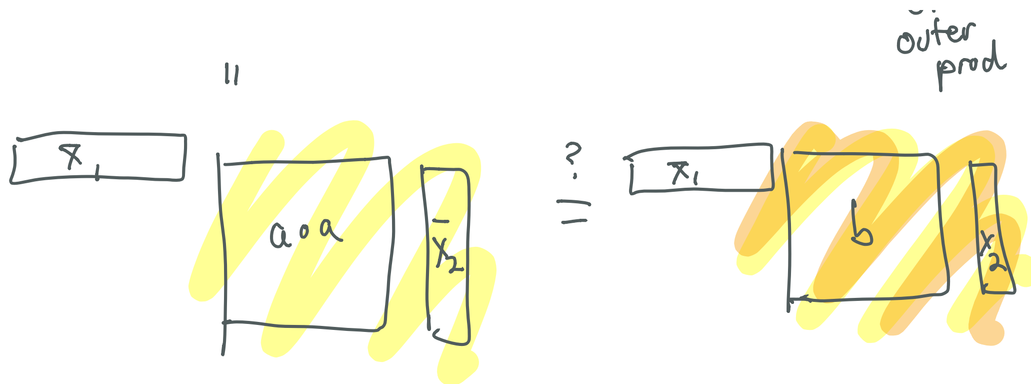
else $\boxed{\text{if } b \neq a \circ a}$

with what prob?

$$A(\bar{x}_1) \cdot A(\bar{x}_2) \stackrel{?}{=} B(\bar{x}_1 \circ \bar{x}_2)$$



by def of



if $a \circ a \neq b$ then

$$\Rightarrow \Pr_{x_2}\left[(a \circ a)\, x_2 \neq b \cdot x_2\right] \geq 1/2$$

$$\Pr_{x_1, x_2}\left[x_1 \cdot (a \circ a) \cdot x_2 \neq x_1 \cdot b \cdot x_2\right] \geq \frac{1}{2} \cdot \frac{1}{2}$$

$$\geq 1/4$$

so test fails with prob $\geq 1/4$

$\square$

Similar argument for if $c \neq a \circ b$
then test fails
with const prob

$\Rightarrow$ if test passes
all three proofs encode
same assignment a

How do we know that a is a $\underline{\text{sat}}$ assignment?

Satisfiability Test:

pick $r \in_R \mathbb{Z}_2^n$

Compute Γ, α_i's, β_{ij}'s, γ_{ijk}'s ⟵ fctns of r & coeffs ~~of~~ polys from CNF clauses

$\downarrow$ x_i's, $\downarrow$ y_{ij}'s, $\downarrow$ z_{ijk}'s

query proof to get

$SC\text{-}\tilde{A}(\alpha_1, \ldots \alpha_n)$ to get w_0

$SC\text{-}\tilde{B}(\beta_{11} \ldots \beta_{nn})$ " " w_1

$SC\text{-}\tilde{C}(\gamma_{111} \ldots \gamma_{nnn})$ " w_2

Verify

$$0 = \Gamma + w_0 + w_1 + w_2 \pmod 2$$

hopefully means $\sum r_i \hat{C}_i(a) = 0$

PCP theorems $\Rightarrow$ hardness of approximation theorems

<u>def</u> $L \in PCP_{1,s}[r,q]$

$\exists$ prob poly time V
tosses r coins
queries q bits

s.t. $x \in L \Rightarrow \exists \pi$ s.t. V accepts with prob 1

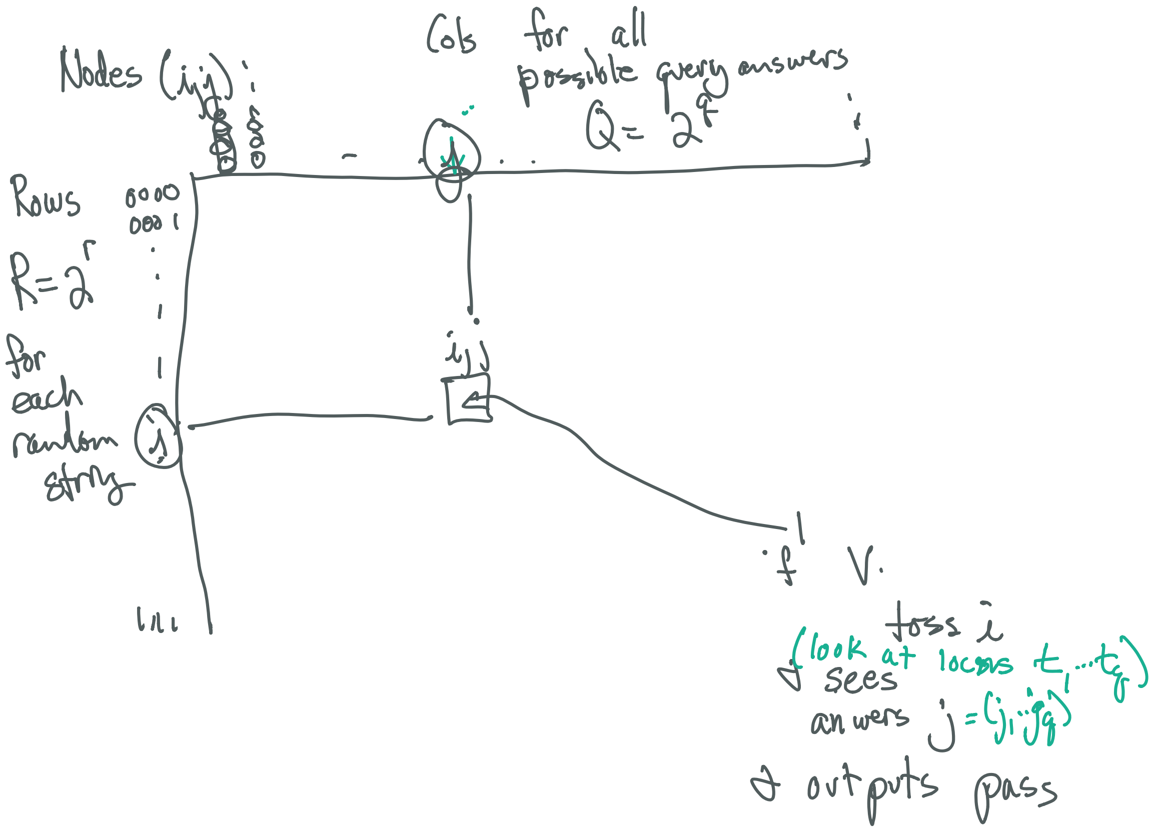
$x \notin L \Rightarrow \forall \pi$, V accepts with prob $\leq s$

(assume V is "non adaptive"
picks all queries before looking at answers)

FGLSS graph:

for a given input x

have a large clique iff $x \in L$



Construct G:

node ~ entry in table of values

edge ~ entries that are consistent

$t_1 \dots t_q$ & $t_1' \dots t_q'$

if they look at same locn get same answer

1) no edge bet M_{ij} & $M_{ij'}$ for $j \neq j'$

$\Rightarrow$ any clique in G
has ≤ 1 node per row

2) if 2 rows query disjoint bits
have complete bipartite
graph between their
nodes

3) clique corresponds to partial proofs
size clique $\geq$ # random choices
for verifier to
accept
$\geq$ (prob of accept) $\cdot 2^r$

if $3SAT \in PCP_{1,s}(r,q)$

then $\phi \in 3SAT \Rightarrow$ pick cols consistent
with π (good proof)
cause every row to pass

~~some~~

all passing rows consistent

$\Rightarrow$ clique size $\geq 2^r$

if $\phi \notin 3SAT$:

$\omega(G) \leq$ ~~[illegible]~~ $s \cdot 2^r$

else, $\exists$ proof consistent with $> s \cdot 2^r$ rows that convinces V.