Today:

Undirected ST connectivity

Linear Algebra + random walks

Saving random bits via random walks.

Recall from last time:

<u>Thm</u> Cover time of undirected graph is $O(n^3)$

S-T connectivity (UST-Conn)

Input: Undirected G, nodes s, t

Output: "Yes" if s & t connected

"No" o.w.

Can solve in poly time, in many ways.

What about small space?

RL = class of problems solvable by randomized log-space computations

[no charge for input space (read only), but can only have const # ptrs ...]

Thm UST-Conn $\in$ RL

Algorithm:

start at s

take random walk for n^3 steps

if ever see t, output "Yes"

o.w. output "No"

Complexity:

Keep track of # steps so far — logspace

edges at each node & toss coin to pick one randomly

Behavior:

If s,t not connected, never output "yes"

If s,t connected

$$h_{st} \le C_s(G_s) \le n^3$$

connected component of s

$$\Pr[\text{output "no"}] = \Pr[\text{start at } s,\ \text{walk} \ge c \cdot E[C_s(G_s)] \text{ steps} + \text{don't see } t]$$

$$\le \frac{1}{c} \quad \text{by Markov's} \neq \quad \blacksquare$$

Comments

- Actually USTCONN $\in L$!!!
- Open is $RL = L$?

  we know $RL \in L^{3/2}$

Linear Algebra Review

def v is an eigenvector of A with corresponding eigenvalue λ iff $vA = \lambda v$

def $\mathcal{L}_2$-norm of $v = (v_1 \dots v_n) = \sqrt{\sum_{i=1}^{n} v_i^2}$

def $v^{(1)} \dots v^{(m)}$ orthonormal if

$$\underbrace{v^{(i)} \cdot v^{(j)}}_{\text{inner product}} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{o.w.} \end{cases}$$

normal

orthogonal

inner product $\sum_{\ell} v^{(i)}(\ell) \cdot v^{(j)}(\ell)$

example: P = transition matrix of d-reg undirected graph (doubly stochastic)

$$\left(\tfrac{1}{n} \dots \tfrac{1}{n}\right) \cdot P = 1 \cdot \left(\tfrac{1}{n} \dots \tfrac{1}{n}\right)$$

also: $\left(\tfrac{1}{\sqrt{n}} \dots \tfrac{1}{\sqrt{n}}\right) \cdot P = 1 \cdot \underbrace{\left(\tfrac{1}{\sqrt{n}} \dots \tfrac{1}{\sqrt{n}}\right)}_{\mathcal{L}_2\text{-norm} = 1}$

$\mathcal{L}_2$-norm $= 1$
⫴
normal

Just like Lake Wobegon, where all the children are above-average

In this class, all theorems are important

"Important" Thm: transition matrix P real + symmetric

doubly stochastic e.g. r.w. on undir graph d-reg

$\Rightarrow \exists$ e-vectors $v^{(1)} \dots v^{(n)}$ with corresponding forming orthonormal basis e-values $1 = \lambda_1 \geq |\lambda_2| \geq \dots \geq |\lambda_n|$

when d-regular graph

$+ \; v^{(1)} = \frac{1}{\sqrt{n}}(1 \dots 1)$ ← set so that $\|v^{(1)}\|_2 = 1$

Assume P has all positive entries & has e-vecs $v^{(1)} \dots v^{(n)}$ with corresponding evals $\lambda_1 \dots \lambda_n$

Fact

(1) αP has evecs $v^{(1)} \dots v^{(n)}$ with corresp evals $\alpha\lambda_1, \dots, \alpha\lambda_n$

(2) $P+I$ " " " " " " $\lambda_1+1, \dots, \lambda_n+1$

(3) P^k " " " " " " $\lambda_1^k, \dots, \lambda_n^k$

(4) P stochastic $\Rightarrow$ $|\lambda_i| \le 1$ $\forall i$

why?

(1) $vP = \lambda v \iff v\alpha P = \lambda \alpha P$

(2) $v(P+I) = vP + vI = \lambda v + v = (\lambda+1) v$

self-loops: $\frac{P+I}{2}$ = "stay put with prob ½ & walk with prob ½"

(3) $v P^k = (vP) P^{k-1} = \lambda v P^{k-1} = \lambda^2 v P^{k-2} = \dots = \lambda^k v$

k-step walks

(4) For all i, let $I = \{ j \mid v_j^{(i)} > 0 \}$

then
$$\lambda \sum_{j \in I} v_j^{(i)} = \sum_{j \in I} \sum_k v_k^{(i)} P_{kj}$$
$$\le \sum_{\substack{j,k \\ \text{st } j,k \in I}} v_k^{(i)} P_{kj}$$
$$\le \sum_{k \in I} v_k^{(i)} \underbrace{\sum_{j \in I} P_{kj}}_{\substack{\le 1 \\ \text{since} \\ \text{stochastic}}} \le \sum_{k \in I} v_k^{(i)}$$

$\therefore \lambda \le 1$

Note: if $v^{(1)} \ldots v^{(n)}$ orthonormal basis then **any** vector w is expressible as linear combination of $v^{(i)}$'s

$$w = \sum \alpha_i v^{(i)}$$

& L_2 norm of w is $\underbrace{\sqrt{\sum \alpha_i^2}}_{\text{why?}}$

$$\|w\|_2 = \sqrt{\sum \alpha_i v^{(i)} \cdot \sum \alpha_j v^{(j)}}$$

$$= \sqrt{\sum \alpha_i \alpha_j \underbrace{v^{(i)} \cdot v^{(j)}}} \quad \begin{cases} = 0 & \text{if } i \neq j \\ = 1 & \text{if } i = j \end{cases}$$

$$= \sqrt{\sum \alpha_i^2}$$

Mixing times

How long does it take to reach stationary distribution?

def. $\varepsilon > 0$

Mixing time, $T(\varepsilon)$, of M.C. A with stationary distribution π is min t s.t. $\forall\ \pi^{(0)}$, $\|\pi - \pi^{(0)} A^t\|_1 < \varepsilon$

def M.C. A is rapidly mixing if $T(\varepsilon) = \text{poly}(\log n, \log 1/\varepsilon)$

n = # states

examples: r.w. on complete graph, random graph

Thm P is transition matrix of undirected, nonbipartite,* d-regular connected graph $\Leftarrow$ ust

if bipartite, then $\lambda_2 = 1$

π_0 is start dist

π is stationary dist $= (\frac{1}{n}, \dots, \frac{1}{n})$ so $\pi P = \pi$

Then $\| \pi_0 P^t - \pi \|_2 \le |\lambda_2|^t$ $\Leftarrow$ exponentially decreasing distance if $1 - \lambda_2 =$ constant

Proof

P real, symmetric $\Rightarrow$ evecs $v^{(1)} \dots v^{(n)}$ are orthonormal basis with evals $1 = \lambda_1 \ge |\lambda_2| \ge \dots \ge |\lambda_n|$

so any vector, in particular, π_0, can be expressed as linear combination of $v^{(i)}$'s

$$\pi_0 = \sum_{i=1}^{n} \alpha_i v^{(i)}$$

$$\text{so } \pi_0 \cdot P^t = \sum_{i=1}^{n} \alpha_i \underbrace{v^{(i)} P^t}_{= \lambda_i^t v^{(i)}}$$

$$= \alpha_1 \underbrace{\lambda_1^t}_{\equiv 1} v^{(1)} + \alpha_2 \lambda_2^t v^{(2)} + \dots$$

what is α_1?

$$v^{(1)} = \frac{1}{\sqrt{n}} (1 \dots 1)$$

$$\pi^{(0)} \cdot v^{(1)} = \alpha_1 \underbrace{v^{(1)} \cdot v^{(1)}}_{=1} + \sum \alpha_i \lambda_i^t \underbrace{v^{(i)} \cdot v^{(1)}}_{=0}$$

$$\frac{1}{\sqrt{n}} \underbrace{\pi^{(0)} (11 \dots 1)}_{=1} = \pi^{(0)} \cdot v^{(1)} = \alpha_1$$

$$\frac{1}{\sqrt{n}} = \alpha_1$$

$$\Rightarrow \alpha_1 \cdot v^{(1)} = \frac{1}{n} \cdot (1 \dots 1)$$

then $\| \pi_0 P^t - \underbrace{\alpha_1 v^{(1)}}_{} \|_2 = \| \sum_{i=2}^{n} \alpha_i \lambda_i^t v^{(i)} \|_2$

$|\lambda_2|^t$ goes to 0 so has to be stationary

$$= \sqrt{\sum_{i=2}^{n} \alpha_i^2 \lambda_i^{2t}}$$ previous note

$$\le |\lambda_2|^t \sqrt{\sum_{i=2}^{n} \alpha_i^2}$$ since $|\lambda_2| \ge |\lambda_3| \ge \dots$

$$\le |\lambda_2|^t \, \| \pi_0 \|_2$$ by Note on previous page & since $\sum_{i=0}^{1} \alpha_i^2 \ge 0$

$$\le |\lambda_2|^t$$ since $L_2 \le L_1 = 1$ when entries ≤ 1

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