6.842 Randomness and Computation	April 2, 2020
Homework 4	
Lecturer: Ronitt Rubinfeld	*Due Date: April 14, 2020*

Homework guidelines: You may work with other students, as long as (1) they have not yet solved the problem, (2) you write down the names of all other students with which you discussed the problem, and (3) you write up the solution on your own. No points will be deducted, no matter how many people you talk to, as long as you are honest. If you already knew the answer to one of the problems (call these "famous" problems), then let me know that in your solution writeup – it will not affect your score, but will help me in the future. It's ok to look up famous sums and inequalities that help you to solve the problem, but don't look up an entire solution.

1. Give a *deterministic* poly(n)-time algorithm that, given n, finds a coloring of the edges of the complete graph K_n by two colors, such that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.

2. **(Edge expansion)** An n-vertex d-regular graph $G = (V, E)$ is called an (n, d, ρ)-edge expander if for every subset S of vertices satisfying $|S| \leq n/2$,

$$|E(S, \overline{S})| \geq \rho d|S|$$

where $E(S, T)$ denotes the set of edges $(u, v) \in E$ with $u \in S$ and $v \in T$.

Prove that for every n-vertex d-regular graph, there exists a subset S of $\leq n/2$ vertices such that

$$E(S, \overline{S}) \leq \frac{dn}{4}(1 + \frac{1}{n-1})$$

(Hint: use the probabilistic method). Conclude that there does not exist an (n, d, ρ)-edge expander for any constant $\rho > 1/2$: more formally, for $\rho > 1/2$, there exists n_0 such that for all d and $n > n_0$, there is no (n, d, ρ) edge expander.

3. **(Random bipartite graphs are good vertex expanders)** A graph $G = (V, E)$ is called an (n, d, c)-vertex expander if it has n vertices, the maximum degree of a vertex is d and for every subset $W \subseteq V$ of cardinality $|W| \leq n/2$, the inequality $|N(W)| \geq c|W|$ holds, where $N(W)$ denotes the set of all vertices in $V \setminus W$ adjacent to some vertex in W. By considering a random bipartite 3-regular graph on $2n$ vertices obtained by picking 3 random permutations between the 2 color classes, one can prove that there exists $c > 0$ such that for every n there exists a $(2n, 3, c)$-vertex expander.

For this homework, just prove that for any subset L of size at most $n/2$ of the "left vertices", (with constant probability) there are at least $(1 + \epsilon)|L|$ "right" neighbors.

It is fine to allow multiple edges in the construction.

Should prove: For all $p > 0$ and all $0 < \epsilon < 1$, there exists $\gamma > 0$ so that for any n, with probablity at least $1 - p$, a random graph constructed using the way described satisfies that any subset K of the left vertices of size $<= \gamma n$ have $>= (1 + \epsilon)|K|$ vertices.