Sub-linear Algorithms	April 6, 2017
Homework 6	
Lecturer: Ronitt Rubinfeld	*Due Date: April 13, 2017*

Homework guidelines: You may work with other students, as long as (1) they have not yet solved the problem, (2) you write down the names of all other students with which you discussed the problem, and (3) you write up the solution on your own. No points will be deducted, no matter how many people you talk to, as long as you are honest. It's ok to look up famous sums and inequalities that help you to solve the problem, but don't look up an entire solution.

1. A *vertex cover* V' of a set of edges E' is a set of nodes such that every edge of E' is adjacent to one of the nodes in V'.

   For graph $G = (V, E)$, let the *transitive closure graph* $TC(G)$ be the graph $G^{(tc)}(V, E^{(tc)})$ where $(u, v) \in E^{(tc)}$ if there is a directed path from u to v in G.

   Let $f : V \to \{0, 1\}$ be a labeling of the vertices of a known directed acyclic graph G by 0 and 1. For any pair of nodes x and y, we say that $x \leq_G y$ if there is a path from x to y in G. We say that f is *monotone* if for all $x \leq_G y$, $f(x) \leq f(y)$. The *minimum distance of f to monotone* is the minimum number of nodes that must be relabeled in order to turn f into a monotone function.

   Let E' be the set of violating edges in $TC(G)$ according to f. Show that the minimum distance of f to monotone is equal to the minimum size of a vertex cover of E'.

2. This problem is about testing monotonicity of functions defined over a directed graph G. The function maps nodes into binary values (i.e., $f : V \to \{0, 1\}$), and we say that it is *monotone* if for all directed edges (u, v), we have that $f(u) \leq f(v)$. We say that f is ϵ-close to monotone if there is a monotone function g such that g and f differ on at most $\epsilon|V|$ entries.

   (a) Let $V = \{v_1, \ldots, v_n\}$. For each directed graph $G = (V, E)$, let $B_G = (V', E')$ be the bipartite graph where $V' = \{v_1, \ldots, v_n\} \bigcup \{v'_1, \ldots, v'_n\}$, and $(v_i, v'_j) \in E'$ iff v_j is reachable from v_i in G.

   Show that a q-query testing algorithm for B_G with distance parameter $\epsilon/2$ yields a q-query testing algorithm for G with distance parameter ϵ.

   (b) Let f be a function on V which is ϵ-far from monotone over graph G. Then $TC(G)$ has a matching of violated edges of size at least $(\epsilon/2)|V|$. (Recall previous problem).

   (c) Show that if f is a function over bipartite graph G, there is a test for monotonicity of f with query complexity at most $O(\sqrt{|V|/\epsilon})$.