Lecture 18

More Boosting

Weak Learning

def. algorithm $\mathcal{A}$ weakly PAC learns concept class $\mathcal{C}$ if $\exists\, \gamma > 0$ s.t.

$\forall\, c \in \mathcal{C}$ + $\forall$ dists $\mathcal{D}$,

given examples of c according to $\mathcal{D}$

$\mathcal{A}$ outputs h s.t. $\Pr_{\mathcal{D}}[h(x) \neq c(x)] \leq \frac{1}{2} - \frac{\gamma}{2}$

$\uparrow$ advantage

Thm if $\mathcal{C}$ can be weakly PAC learned (on any $\mathcal{D}$) then $\mathcal{C}$ can be (strongly) PAC learned.

Weak vs. Strong Learning

Def. Algorithm $\mathcal{A}$ weakly "PAC learns" concept class $\mathcal{C}$

if $\forall\ c \in \mathcal{C}$ & $\forall$ dists $\mathcal{D}$ $\quad \exists\ \gamma > 0$

$\forall\ \cancel{\varepsilon}, \delta > 0$ $\quad$ ($\delta = \frac{1}{4}$ or $\frac{1}{n^2}$ doesn't affect)

with prob $\geq 1-\delta$

given examples of c

$\mathcal{A}$ outputs h s.t. $\Pr_{\mathcal{D}}[h(x) \neq c(x)] \leq \cancel{\varepsilon}\ \frac{1}{2} - \frac{\gamma}{2}$

↑ advantage

It was conjectured that distribution free weak learning was really weaker but surprise!

Can "boost" a weak learner

Thm if $\mathcal{C}$ can be weakly learned on <u>any</u> dist $\mathcal{D}$ then $\mathcal{C}$ can be (strongly) learned.

Applications

1) "theoretical"

- Unif dist Algorithms for poly term DNF
  weight w - poly threshhold fctns } low degree alg doesn't work well

  ∴ (Boosting + KM)

- Ave case vs. worst case

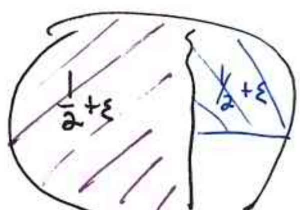
2) practical - Boosting
   Freund-Schapire

Good & Bad Ideas

1) simulate weak learner several times on same distribution & take majority answer
   -or-
   best answer

   gives better confidence
   but doesn't reduce error, what if always get same answer?

2) filter out examples on which current hypothesis does well & run weak learner on part where you do badly.



Problem: given a new example, how do you know which section it is in?

3) Keep some samples on which you are ok

alway use majority vote on all previous hypotheses to predict value of new samples

history: Schapire, Freund-Schapire, Impagliazzo-Servedio. Klivans

Filtering Procedures

- decide which samples to keep, which to throw out
- samples on which so far you guess correctly ← need for checking future hypotheses
  incorrectly ← need to improve on these

The setting

- Given labelleled examples

  $(x_1, f(x_1)), (x_2, f(x_2)), \ldots$

  $x_i \in_R \mathcal{D}$

  $f \in \mathcal{C}$

- Given weak learning alg WL which weakly learns (advantage $\frac{\gamma}{2}$) on <u>any</u> dist $\mathcal{D}'$

Boosting Algorithm

- Stage 0 (Initialize)

  $\mathcal{D}_0 \leftarrow \mathcal{D}$

  run WL on $\mathcal{D}_0$ to generate (whp)

  C_1 s.t. $\Pr_{\mathcal{D}_0}[f(x) = C_1(x)] \geq \frac{1}{2} + \gamma/2$

- For $i = 1 \ldots T = O(\frac{1}{\gamma^2 \varepsilon^2})$ stages, stage i: (can stop if Majority$(C_1 \ldots C_i)$ correct on $\geq 1-\varepsilon$ inputs)

  (1) Construct $\mathcal{D}_i$ via "filtering procedure":

  { favor pts on which maj of $C_1 \cdots C_i$ don't do well

  but also keep some other points }

  Will specify soon

  (2) run WL on examples from $\mathcal{D}_i$ to output

  C_{i+1} s.t. $\Pr_{\mathcal{D}_i}[f(x) = C_{i+1}(x)] \geq \frac{1}{2} + \frac{\gamma}{2}$

- output $C = MAJ(C_1 \ldots C_T)$

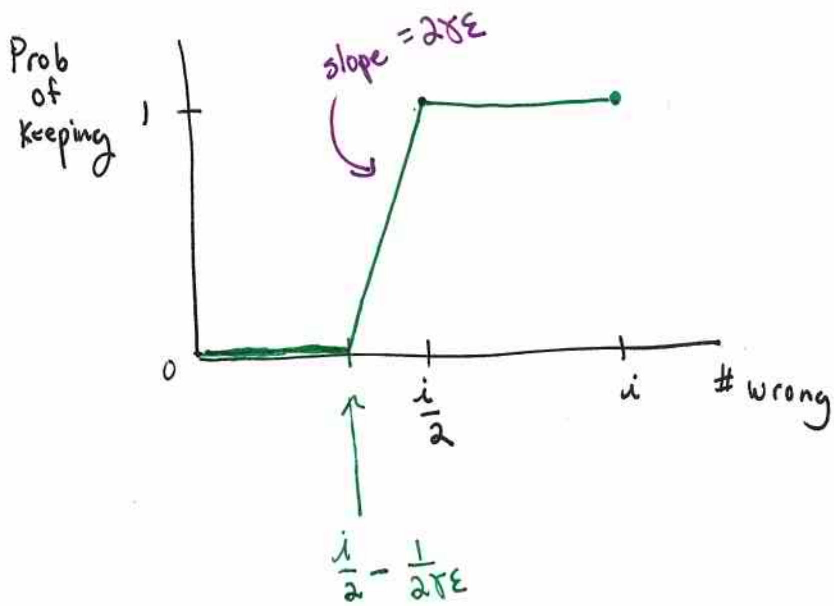
Filtering procedure

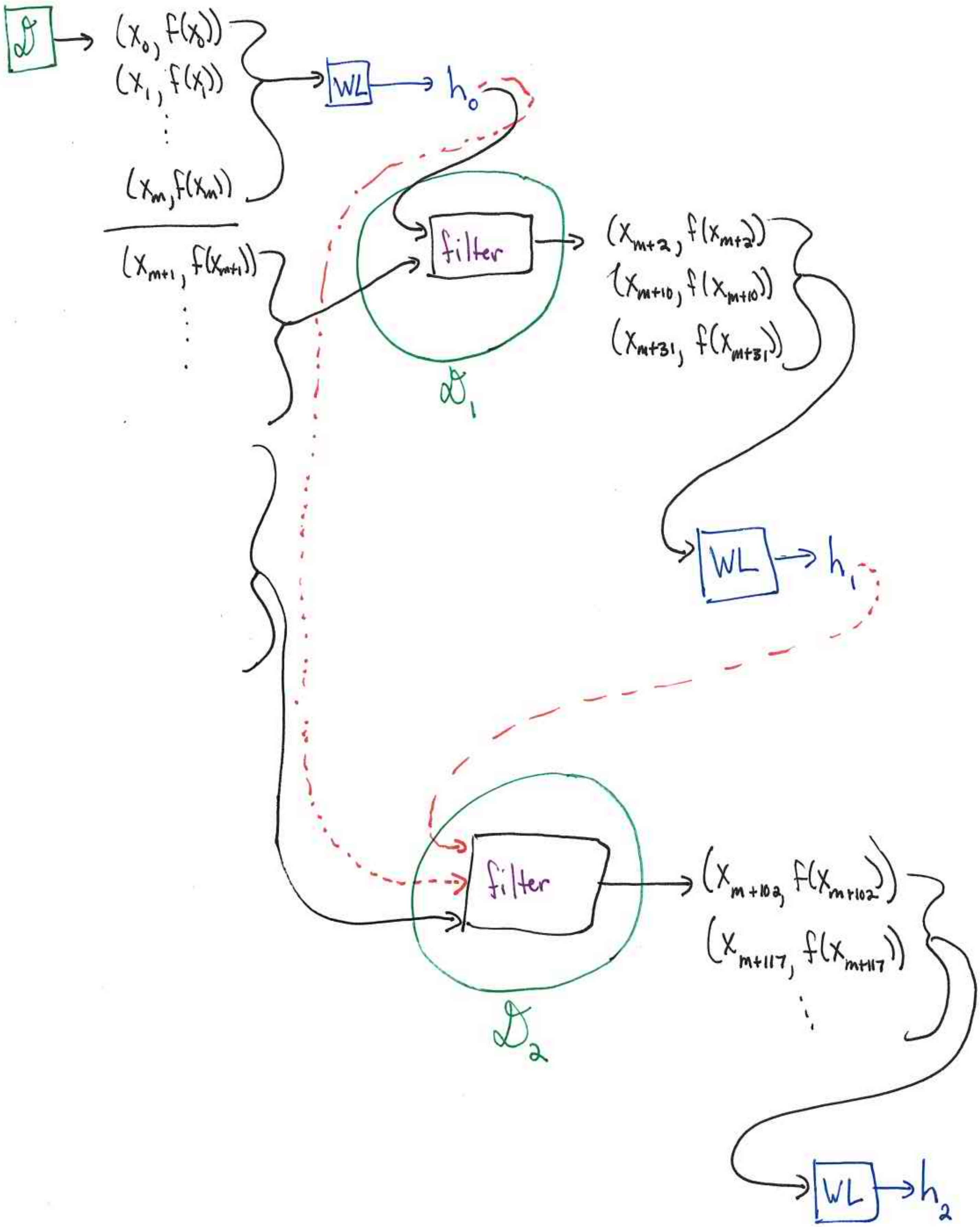
Given new example $x, f(x)$ from example oracle

- if majority of $c_1 \ldots c_i$ wrong, Keep it
  ie. $\geq \frac{i}{2}$

- if large majority right, then discard
  ie. $\#\text{right} - \#\text{wrong} > \frac{1}{\gamma\varepsilon}$
  or $\#\text{wrong} \leq \frac{i}{2} - \frac{1}{2\gamma\varepsilon}$

- else $\#\text{right} - \#\text{wrong} = \frac{\alpha}{\gamma\varepsilon}$ for $0 < \alpha < 1$
  $\#\text{wrong} - \#\text{right} = \frac{-\alpha}{\gamma\varepsilon}$
  So Keep with prob $= 1 - \alpha$




D
(x_0, f(x_0))
(x_1, f(x_1))
(x_m, f(x_m))
(x_{m+1}, f(x_{m+1}))
WL
h_0
filter
D_1
(x_{m+2}, f(x_{m+2}))
(x_{m+10}, f(x_{m+10}))
(x_{m+31}, f(x_{m+31}))
WL
h_1
filter
D_2
(x_{m+102}, f(x_{m+102}))
(x_{m+117}, f(x_{m+117}))
WL
h_2

Need to show:

1) Output ~~is~~ has nontrivial agreement with f

2) # samples needed not too bad

why could it be bad? if throw out lots of samples, might need to wait a long time before WL can give an output.

but if throw out too many samples then you already have a good hypothesis!

will stop if $Maj(c_1 \ldots c_i)$ correct on $\geq 1-\varepsilon$ fraction of inputs

o.w. $Maj(c_1 \ldots c_i)$ incorrect on $> \varepsilon$ fraction so filtering procedure outputs sample with prob $\geq \varepsilon$ (+ in expectation, every $1/\varepsilon$ samples of $\mathcal{D}$ at least one makes it thru the filtering system)

$\Rightarrow$ filtering slows down sample collection by $\leq O(1/\varepsilon)$

So lets focus on (1)

Notation

$$\cdot\ R_c(x) = \begin{cases} +1 & \text{if } f(x) = c(x) \\ -1 & \text{if } f(x) \neq c(x) \end{cases}$$

"is c correct on x?"

$$\cdot\ N_i(x) = \sum_{1 \le j \le i} R_{c_i}(x)$$

after iteration i, how many c's correct? (#right − #wrong)

$$\cdot\ M_i(x) = \begin{cases} 1 & \text{if } N_i(x) \le 0 \\ 0 & \text{if } N_i(x) \ge \frac{1}{\varepsilon \gamma} \\ 1 - \varepsilon \cdot \gamma \cdot N_i(x) & \text{o.w.} \end{cases}$$

prob of keeping x in filtering (after stage i)

note – all "wrong" x included in M
also some "right" x included

Note that new distribution on samples is proportional to M_i:

$$\cdot\ D_{M_i}(x) \equiv \frac{M_i(x)}{\sum_x M_i(x)}$$

distribution induced by M

note $D_{M_i}(x) = \mathcal{D}_i$

$\sum_x M_i(x)$ includes all "wrong" x but also x for which maj that isn't overwhelming are correct } upper bounds # wrong x

How correct are we w.r.t. D_{M_i}?

$$\cdot\ Adv_c(M_i) = \sum_x R_c(x)\, M_i(x)$$

"Advantage" of c on M
$\sim Pr[\text{correct}] - Pr[\text{incorrect}]$
$= 2 \cdot Pr[\text{correct}] - 1$

$$\cdot\ Pr_{x \in D_{M_i}}[c(x) = f(x)] \equiv \frac{1}{2} + \underbrace{\frac{Adv_c(M_i)}{2 \cdot \sum_x M_i(x)}}_{\gamma/2}$$

Boost ⑧
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Note:

if $\sum M_i(x) \geq \varepsilon 2^n$

$$Adv_c(M_i) \geq \gamma \cdot \varepsilon \cdot 2^n$$

convert claim about WL ⇒ claim about advantage
i.e. if have γ advantage on output of WL
& still almost wrong on lots of inputs
then new advantage is pretty good

if not, then you are done

Begin Proof

For input x

let $A_i(x) \leftarrow \sum_{0 \leq j \leq i-1} R_{c_{j+1}}(x)\, M_j(x)$

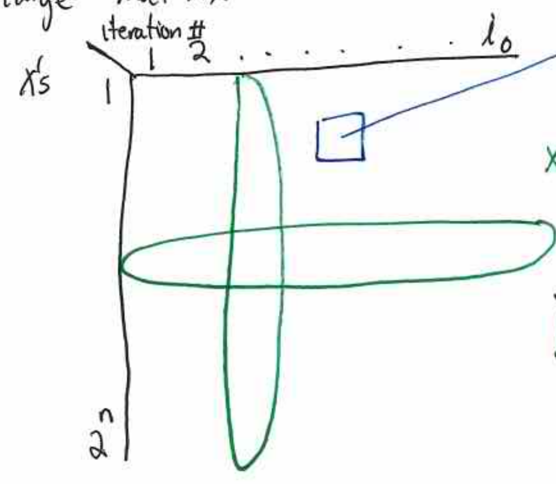
strange - indices don't match
$c_1 \ldots c_j$ define $\mathcal{D}_j$
but c_{j+1} learned from WL on $\mathcal{D}_j$

Claim $A_i(x) \leq \frac{1}{\varepsilon\gamma} + \frac{\varepsilon\gamma}{2} \cdot i$

- bounds advantage per input
- only helps after $\frac{1}{\varepsilon\gamma}$ rounds

Plan for use of claim:

Consider large matrix:



$(k,j)^{th}$ Entry: $R_{c_{j+1}}(x_k)\, M_j(x_k)$

x's row sum $= \sum_{0 \leq j \leq i_0} R_{c_{j+1}}(x)\, M_j(x) = A_{i_0+1}(x)$

j^{th} col sum $= \sum_x R_{c_{j+1}}(x)\, M_j(x)$

$= Adv_{c_{j+1}}(M_j) \leftarrow \geq \gamma \cdot \varepsilon \sum_x M_j(x)$ else algorithm stops

Goal: lower/upper bound average entry in matrix

lower bound:

lower bound average entry in column via

- correctness of WL
- fact that algorithm proceeds

$\Rightarrow$ lots of error

$\Rightarrow \sum_x M_j(x)$ big

$\Rightarrow$ lots of progress in WL in absolute terms

upper bound:

upper bound rows via claim

- if advantage is too much, lose measure

this is where majority rule & weighting scheme is used

More details:

Assume claim, prove theorem:

Assume haven't terminated at i_0+1^{th} stage

- so $\text{error}(C_{i_0}) \geq \varepsilon$

$$\sum_x M_{i_0}(x) \geq \varepsilon 2^n$$

Claim $\Rightarrow$

$$\sum_x A_{i_0+1}(x) = \sum_x \sum_{0 \leq j \leq i_0} R_{c_{j+1}}(x) M_j(x) \quad \text{def of } A_{i_0+1}$$

$$= \sum_{0 \leq j \leq i_0} Adv_{c_{j+1}}(M_j) \quad \text{def of } Adv_{c_{j+1}}$$

$$\geq \underbrace{(\gamma \varepsilon 2^n)}_{\text{from "note"}} (i_0+1)$$

$$+ \quad \sum_x A_{i_0+1}(x) \leq \sum_x \left(\frac{1}{\varepsilon\gamma} + \frac{\varepsilon\gamma}{2} \cdot (i_0+1) \right) \quad \text{claim}$$

$$= 2^n \left(\frac{1}{\varepsilon\gamma} + \frac{\varepsilon\gamma}{2} (i_0+1) \right)$$

putting together:

$$(\varepsilon\gamma)(i_0+1) \leq \frac{1}{\varepsilon\gamma} + \frac{\varepsilon\gamma}{2}(i_0+1)$$

so $\quad \frac{\varepsilon\gamma}{2}(i_0+1) \leq \frac{1}{\varepsilon\gamma} \Rightarrow i_0 \leq \frac{2}{\varepsilon^2\gamma^2} - 1$ ∎

Proof of claim:

Question: how can an input add to cumulative advantage througout algorithm?

Observations:

- if algorithm's hypotheses $c_1 \ldots c_i$ are overwhelmingly correct on x, then not at all because x gets measure 0
- if algorithm's hypotheses are doing badly (mostly wrong) then not at all because they decrease advantage
- Main Issue:

  can wander in middle - $\underbrace{\text{majority}}_{\text{increase advantage}}$ correct but $\underbrace{\text{not large majority}}_{\text{so have positive measure}}$

need to bound this case.

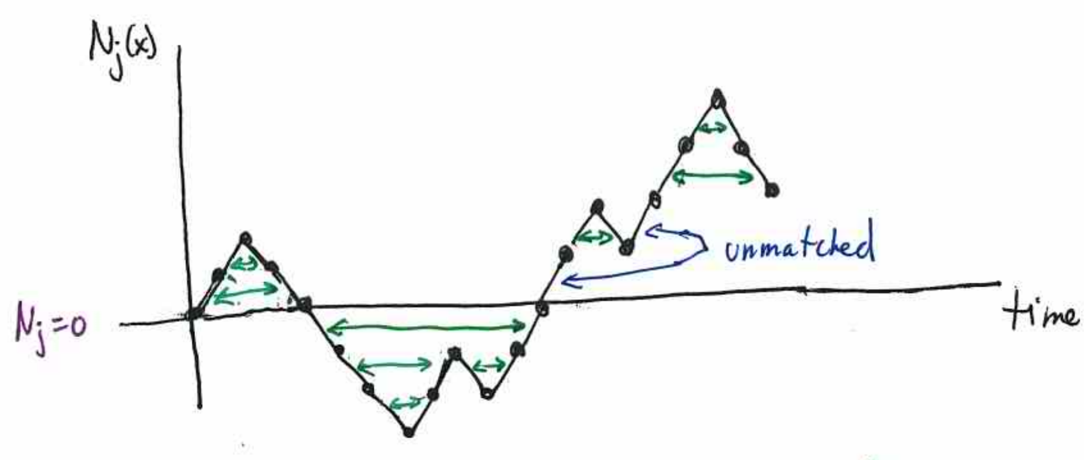
Proof of Claim

First, strange but obvious fact:

Fact "elevator argument"
If one spends any amount of time in an elevator, then no matter what sequence of buttons pushed, one ascends from k^{th} to $k+1^{st}$ floor at most one more time than one descends from the $k+1^{st}$ to k^{th} floor.
(analogous for negative floors $-k$ & $-(k+1)$)

Proof by picture:

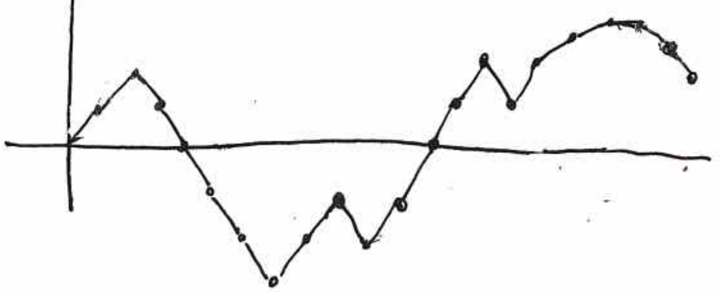
for any x



match transitions going up with those going down on same level (as in parentheses) but what is behavior of $\sum_{j<k} \underbrace{R_{c_{j+1}}}_{\pm 1}(x) \underbrace{M_j(x)}_{\in[0,1]}$?

$\Rightarrow$ |slope| ≤ 1 (in fact, $\leq 2\gamma\varepsilon$) + same sign as $N_j(x)$

$\sum_{j<k} R_{c_{j+1}}(x) M_j(x)$



$$\boxed{\text{Recall: } A_i \equiv \sum_{0\leq j\leq i-1} R_{c_{j+1}}(x)\, M_j(x)}$$

Matching:

For $k \geq 0$:

match $a = j$ s.t. $N_j(x) = k$ + $N_{j+1}(x) = k+1$

with $b = j'$ s.t. $N_{j'}(x) = k+1$ + $N_{j'+1}(x) = k$

For $k<0$: analogously match $-k$ to $-(k+1)$
with $-(k+1)$ to $-k$

For each matched pair:

Will bound contribution from matched pairs
by $\varepsilon\gamma$ per pair using bound on slope
(and total of $\frac{\varepsilon\gamma i}{2}$)

(for each matched pair (a,b) cont.)

just by assumption that $R_{c_{a+1}}(x) = +1$ & $R_{c_{b+1}}(x) = -1$

$$\underbrace{R_{c_{a+1}}(x)}_{\substack{+1 \\ \text{elevator} \\ \text{going up}}} \; \underbrace{M_a(x)}_{N_a(x)=k} + \underbrace{R_{c_{b+1}}(x)}_{\substack{-1 \\ \text{elevator} \\ \text{going} \\ \text{down}}} \; \underbrace{M_b(x)}_{N_b(x)=k+1} = M_a(x) - M_b(x)$$

if $\quad 0 \le k \le \frac{1}{\varepsilon\gamma} \quad$ or $\quad 0 \le k+1 \le \frac{1}{\varepsilon\gamma}$

then $\quad \underbrace{M_a(x)} - \underbrace{M_b(x)}$

$$= \left(1 - \varepsilon\gamma N_a(x)\right) - \left(1 - \varepsilon\gamma N_b(x)\right)$$

$$= \cancel{1} - \varepsilon\cancel{\gamma}k - \cancel{1} + \varepsilon\gamma(\cancel{k}+1)$$

$$= \varepsilon\gamma$$

else $\quad M_a(x) - M_b(x) = \begin{cases} 1-1 \\ \text{or} \\ 0-0 \end{cases} = 0$

$\therefore$ each pair contributes $\le \varepsilon\gamma$ to sum
$\le \frac{i}{2}$ pairs

$\Big\} \le \frac{i}{2} \cdot \varepsilon\gamma$ total contribution

Contribution from unmatched edges:

either all unmatched N_i's have negative steps
or all have positive steps

if all negative:

R_{c_j}'s all -1

M_j's all $\in [0,1]$

$\therefore$ contribution of $R_{c_{j+1}}(x)\, M_j(x) < 0$

if all positive:

if unmatched N_i's in $[0, \frac{1}{\varepsilon\gamma}]$

- for each $M_j \in [0,1]$, contribution of $R_{c_{j+1}} M_j(x) \le 1$

- at most $1/\varepsilon\gamma$ of these $\Rightarrow$ total contribution $\le \frac{1}{\varepsilon\gamma}$

if unmatched N_i in $[\frac{1}{\varepsilon\gamma}, \dots]$

then $M_j = 0$

$\Rightarrow$ total contribution $= 0$

$\therefore$ Grand total $\le \frac{1}{2} \cdot \gamma\varepsilon \cdot i + \frac{1}{\varepsilon\gamma}$