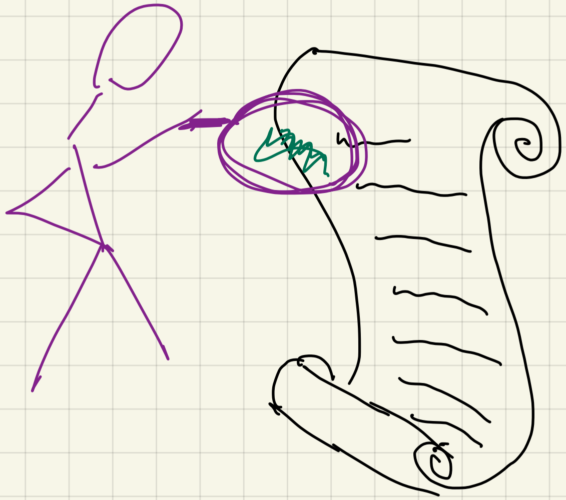
Lecture 23

Probabilistically Checkable Proof Systems
(cont.)



linear fctn : $\forall x, y \quad f(x) + f(y) = f(x+y)$

Self-correcting :

if f is $\frac{1}{8}$ - close to linear g

Do $O(\log \frac{1}{\beta})$ times

Pick y randomly

$\text{answer}_i \leftarrow f(y) + f(x-y)$

Output most common answer_i

then

$\forall x, \ \Pr[\text{output} = g(x)] \geq 1 - \beta$

Self-testing: Given f

Do $O(\frac{1}{\varepsilon})$ times:

Pick x, y randomly

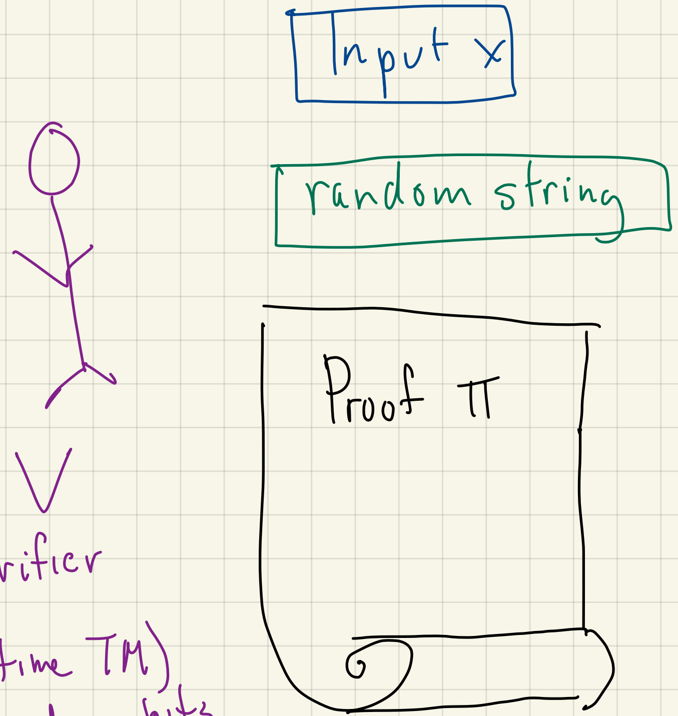
if $f(x) + f(y) \neq f(x+y)$ Fail

Pass

if f linear passes

if f ε-far from linear, fails

Probabilistically Checkable Proofs



← Theorem you want to prove
for today: X is 3CNF
<u>Thm</u> X is satisfiable

verifier
(poly time TM)
using r random bits
+ q queries to proof

Proof: fixed fctn
Verifier can query: what is i^{th} bit?
Charged per query
proof doesn't change based on past questions of verifier
Created by adversary who knows verifier's algorithm + has unlimited computational power

<u>def</u> $L \in PCP(r,q)$ if $\exists\, V$ (ptime TM) s.t.

1) $\forall x \in L \quad \exists\, \pi$ s.t. $\Pr_{V\text{'s random string}}[V, \pi \text{ accepts}] = 1$

2) $\forall x \notin L \quad \forall \pi' \quad \Pr_{V\text{'s random strings}}[V, \pi' \text{ accepts}] \leq 1/4$

e.g. $SAT \in PCP(0, n)$ ← proof settings of all n vars
V doesn't need any randomness

Today: $NP \subseteq PCP(O(n^3), O(1))$ ← crazy?

Actually: $NP \subseteq PCP(O(\log n), O(1))$

Let's start with a "warmup":

$x \cdot y = \sum x_i \cdot y_i$ "inner product"

$\underbrace{x \circ y}_{\text{n-bit vectors}} = \underbrace{(x_1 y_1, x_1 y_2, x_4 y_3, \ldots, x_i y_j, \ldots, x_n y_n)}_{n^2 \text{ bit vector}}$ "outer product"

Fact: if $\underbrace{\bar{a} \neq \bar{b}}_{\text{n-bit vector}}$ then $\Pr_{\bar{r} \in_R \{0,1\}^n}[\bar{a} \cdot \bar{r} \neq \bar{b} \cdot \bar{r}] \geq \frac{1}{2}$

if $\underbrace{A \cdot B \neq C}_{\text{nxn matrices}}$ then $\Pr_{\bar{r}}[\underbrace{A \cdot B \cdot \bar{r}}_{A \cdot (B \cdot \bar{r})} \neq \underbrace{C \cdot \bar{r}}] \geq 1/2$

$A \cdot (B \cdot \bar{r})$ take $O(n^2)$ to compute

also true for "= mod 2"

Fact: if $\bar{a} \neq \bar{b}$ then $\Pr_{\bar{r} \in_R \{0,1\}^n}[\bar{a}\cdot\bar{r} \neq \bar{b}\cdot\bar{r}] \geq \frac{1}{2}$

if $A\cdot B \neq C$ then $\Pr_{\bar{r}}[A\cdot B\cdot\bar{r} \neq C\cdot\bar{r}] \geq 1/2$

Example "application": setting: given vector $\bar{a} = (a_1, a_2, \dots a_n)$

in one step: • can query a_i

• can specify $\bar{y}$ & query $\bar{a}\cdot\bar{y}$

to test if $\bar{a} \equiv (0,0,\dots,0)$:

Do several times:

pick $\bar{r} \in_R \{0,1\}^n$

if $\bar{a}\cdot\bar{r} \neq 0$ output "Fail"

Output PASS

behavior: if $\bar{a} = (0,\dots 0)$ will always PASS

if $\bar{a} \neq (0,\dots 0)$ then FACT $\Rightarrow \Pr_{\bar{r}}[\bar{a}\cdot\bar{r} \neq 0] \geq 1/2$

$\Rightarrow O(1)$ query 0-testing algorithm for n-bit vector in strange model

Arithmetization of 3SAT:

Boolean formula F $\iff$ arithmetic formula $A(F)$ over $\underbrace{\mathbb{Z}_2}_{\text{mod } 2}$

$$T \iff 1$$

$$F \iff 0$$

$$x_i \iff x_i$$

$$\bar{x}_i \iff 1 - x_i$$

$$\alpha \wedge \beta \iff \alpha \cdot \beta$$

$$\alpha \vee \beta \iff 1 - (1-\alpha)(1-\beta)$$

$$\alpha \vee \beta \vee \gamma \iff 1 - (1-\alpha)(1-\beta)(1-\gamma)$$

example: $x_1 \vee \bar{x}_2 \vee x_3 \iff 1 - (1-x_1)\overbrace{(x_2)}^{1-(1-x_2)}(1-x_3)$

Key point: F satisfied by assignment a iff $[A(F)](a) = 1$

$F = \bigwedge C_i$ s.t. $C_i = (y_{i_1} \vee y_{i_2} \vee y_{i_3})$

where $y_{i_j} \in \{ x_1 \dots x_n, \bar{x}_1 \dots \bar{x}_n \}$

$T \Leftrightarrow 1$

$F \Leftrightarrow 0$

$x_i \Leftrightarrow x_i$

$\bar{x}_i \Leftrightarrow 1 - x_i$

$\alpha \wedge \beta \Leftrightarrow \alpha \cdot \beta$

$\alpha \vee \beta \Leftrightarrow 1 - (1-\alpha)(1-\beta)$

$\alpha \vee \beta \vee \gamma \Leftrightarrow 1 - (1-\alpha)(1-\beta)(1-\gamma)$

Consider $C^\sigma(x) = (\hat{C}_1(x), \hat{C}_2(x), \dots)$

s.t. $\hat{C}_i(x) =$ complement of arithmetization of clause C_i

$\Rightarrow$ evaluates to 0 if x satisfies C_i

$\Rightarrow C^\sigma(x) = (0, \dots 0)$ if x satifies F

Observe (1) each $\hat{C}_i$ is deg ≤ 3 poly in x

(2) V knows coeffs of each $\hat{C}_i$

Need to convince V that $C^\sigma(a) = (\hat{C}_1(a), \hat{C}_2(a) \dots) = (0, \dots 0)$ WITHOUT SENDING assignment a

High level idea: special encoding of assignment

Encode satisfiability of F as a collection of polys in vars of assignment
- one for each clause
- eval to 0 if assignment satisfies clause
- low degree
- V knows coeffs - depend on structure of clause & vars of clause.

Note: We are only concerned that V is poly time, here will not be sublinear

note that solving SAT in poly time would be impressive 😉

However, want # queries to proof to be constant

Idea for proof:

- proof contains $\mathcal{C}(a)\cdot r \quad \forall \quad r \in \{0,1\}^n$
- if $\forall i,\ \hat{C}_i(a) = 0,\ \Pr_r[\mathcal{C}(a)\cdot r = 0] = 1$

  if $\exists i$ s.t. $\hat{C}_i(a) \neq 0,\ \Pr[\mathcal{C}(a)\cdot r = 0] = \frac{1}{2}$

  $\Pr[\mathcal{C}(a)\cdot r = 1]$

$F = \bigwedge C_i$ s.t. $C_i = (y_{i_1} \vee y_{i_2} \vee y_{i_3})$

where $y_{i_j} \in \{x_1 \dots x_n, \bar{x}_1 \dots \bar{x}_n\}$

$\mathcal{C}(a) = (\underbrace{\hat{C}_1}_{\text{complement}}(a), \hat{C}_2(a), \dots) \overset{?}{=} (0, 0, \dots 0)$

$\Leftrightarrow$ mod 2 arithmetic

$T \Leftrightarrow 1$

$F \Leftrightarrow 0$

$x_i \Leftrightarrow x_i$

$\bar{x}_i \Leftrightarrow 1 - x_i$

$\alpha \wedge \beta \Leftrightarrow \alpha \cdot \beta$

$\alpha \vee \beta \Leftrightarrow 1 - (1-\alpha)(1-\beta)$

$\alpha \vee \beta \vee \gamma \Leftrightarrow 1 - (1-\alpha)(1-\beta)(1-\gamma)$

What does $\mathcal{C}(a)\cdot r$ look like?

V doesn't know

$$\sum_i r_i \hat{\mathcal{C}}_i(a) = \Gamma + \sum_i a_i \alpha_i + \sum_{i,j} a_i a_j \beta_{ij} + \sum_{i,j,k} a_i a_j a_k \gamma_{ijk} \pmod 2$$

V does know: depend on r_i's + coeffs in $\hat{C}_i$

from here on:

$\alpha_i \to x_i$
$\beta_{ij} \to y_{ij}$
$\gamma_{ijk} \to z_{ijk}$

$\Big\}$ no relation to vars of 3SAT!!!

High level idea: Special encoding of assignment

- proof writes out all linear / deg 2 / deg 3 fctns of assignment

- possible "confusion": "symmetric" for linear case

$$f_x(a) = x \cdot a = A_a(x)$$

inner product

$$(a \circ a) = \begin{pmatrix} a_1a_1 & a_1a_2 & \cdots & a_1a_n \\ a_2a_1 & a_2a_2 & \cdots & \\ \vdots & & \ddots & \\ a_na_1 & \cdots & & a_na_n \end{pmatrix}$$

- for deg 2, 3: $B_a(y) = (a \circ a)^T \cdot y$

$$C_a(z) = (a \circ a \circ a)^T \cdot z$$

A_a, B_a, C_a are all linear fctns $\Rightarrow$ can test linearity & self-correct

Proof can cheat!
- what if A_a, B_a, C_a don't correspond to same assignment?
- is a satisfying?

will write whole fctn their values at single inputs (const#)

only care about

corresponds to V's computation based on coefts of deg 3 polys & r_i's

def

A = all linear fctns evaluated at assignment a

$$A : \mathbb{F}_2^n \to \mathbb{F}_2 \qquad A(x) = \sum a_i x_i = a^T \cdot x$$

V knows x, y, z but not a

B = all deg 2 fctns evaluated at a

$$B : \mathbb{F}_2^{n^2} \to \mathbb{F}_2 \qquad B(y) = \sum_{i,j} a_i a_j y_{ij} = (a \circ a)^T \cdot y$$

C = all deg 3 fctns evaluated at a

$$C : \mathbb{F}_2^{n^3} \to \mathbb{F}_2 \qquad C(y) = \sum_{ijk} a_i a_j a_k z_{ijk} = (a \circ a \circ a)^T \cdot z$$

recall:

$$x \circ y = (x_1 y_1, x_1 y_2, x_1 y_3, \dots, x_i y_j, \dots, x_n y_n)$$

Proof contains:

Complete description of truth tables of $\tilde{A}, \tilde{B}, \tilde{C}$ for all inputs x, y, z

only need value at $x = \alpha, y = \beta, z = \gamma$

extra info helps us check consistency!

$x \circ y = (x_1 y_1, x_1 y_2, x_1 y_3, \ldots, x_i y_j, \ldots, x_n y_n)$

def

A = all linear fctns evaluated at assignment a — $A : \mathbb{F}_2^n \to \mathbb{F}_2$ — $A(x) = \sum a_i x_i = a^T \cdot x$

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Proof contains: **HUGE**

Complete description of truth tables of $\tilde{A}, \tilde{B}, \tilde{C}$ for all inputs x, y, z

only need value at $x = \alpha$, $y = \beta$, $z = \gamma$ but extra info helps us check consistency

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What does verifier need to check in proof?

(1) $\tilde{A}, \tilde{B}, \tilde{C}$ in right form

- all are linear fctns
- correspond to same assignment a

i.e. $\tilde{A}(x) = a^T \cdot x \implies \tilde{B}(y) = (a \circ a)^T \cdot y \implies \tilde{C}(y) = (a \circ a \circ a)^T \cdot z$

Test consistency of self-corrected versions

can only test ε-close to linear but can use self-corrector to access the linear fctns

(2) a is satisfying assignment

- all $\hat{C}_i$'s evaluate to 0 on a

(recall $\mathcal{C}(a) = (\hat{C}_1(a), \hat{C}_2(a), \ldots) \stackrel{?}{=} (0, 0, \ldots 0)$)

complement

$$x \circ y = (x_1 y_1, x_1 y_2, x_4 y_3, \dots, x_i y_j, \dots, x_n y_n)$$

def

A = all linear fctns evaluated at assignment a

$A : \mathbb{F}_2^n \to \mathbb{F}_2 \qquad A(x) = \sum a_i x_i = a^T \cdot x$

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$C : \mathbb{F}_2^{n^3} \to \mathbb{F}_2 \qquad C(y) = \sum_{ijk} a_i a_j a_k z_{ijk} = (a \circ a \circ a)^T \cdot z$

Proof contains:

Complete description of truth tables of $\tilde{A}, \tilde{B}, \tilde{C}$ for all inputs x, y, z

only need value at $x = \alpha$, $y = \beta$, $z = \gamma$ but extra info helps us check consistency

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Test (1) $\tilde{A}, \tilde{B}, \tilde{C}$ in right form: all are linear fctns

Can only test ε-close to linear but can self-correct to access the linear fctns.

- Test $\tilde{A}, \tilde{B}, \tilde{C}$ are all $1/8$-close to linear (i.e. if all linear, PASS; if any one is $1/8$-far FAIL) in $O(1)$ queries
- From now on, use self corrector to get

$sc\text{-}\tilde{A}$, $sc\text{-}\tilde{B}$, $sc\text{-}\tilde{C}$ for all inputs

$\Updownarrow$ a $\qquad \Updownarrow$ b $= a \circ a$? $\qquad \Updownarrow$ c $= a \circ a \circ a$?

use β = prob of getting wrong answer in SC that is so small ($\leq \frac{1}{\text{big enough constant}}$) that union bnd over all queries to $sc\text{-}\tilde{A}$, $sc\text{-}\tilde{B}$, $sc\text{-}\tilde{C}$

$\Rightarrow$ unlikely to ever see "error" in SC def $\quad x \circ y = (x_1 y_1, x_1 y_2, x_1 y_3, \ldots, x_i y_j, \ldots, x_n y_n)$

A = all linear fctns evaluated at assignment a

$$A : \mathbb{F}_2^n \to \mathbb{F}_2 \qquad A(x) = \sum a_i x_i = a^T \cdot x$$

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Test (1) $\tilde{A}, \tilde{B}, \tilde{C}$ in right form:
- all are linear fctns
- correspond to same assignment a

i.e. $\tilde{A}(x) = a^T \cdot x \Rightarrow \tilde{B}(y) = (a \circ a)^T \cdot y \Rightarrow \tilde{C}(z) = (a \circ a \circ a)^T \cdot z$

Test consistency of self corrections

Goal: Pass if $sc\text{-}\tilde{B} = sc\text{-}\tilde{A} \circ sc\text{-}\tilde{A}$

$sc\text{-}\tilde{C} = sc\text{-}\tilde{A} \circ sc\text{-}\tilde{B}$

test $sc\text{-}\tilde{A}$ & $sc\text{-}\tilde{B}$ correspond to same a_i's

Outer Product Tester: Pick random x_1, x_2, x, y

Test $\quad sc\text{-}\tilde{A}(x_1) \cdot sc\text{-}\tilde{A}(x_2) = \sum a_i x_{1i} \circ \sum a_j x_{2j} = \sum_{ij} a_i a_j x_{1i} x_{2j} = \sum b_{ij} x_{1i} x_{2j}$

$$= sc\text{-}\tilde{B}(x_1 \circ x_2)$$

$$sc\text{-}\tilde{A}(x) \cdot sc\text{-}\tilde{B}(y) = \left(\sum a_i x_i\right) \cdot \left(\sum b_{jk} y_{jk}\right) = \sum_{ijk} a_i b_{jk} x_i y_{jk} = \sum a_i a_j a_k x_i y_{jk}$$

$$= sc\text{-}\tilde{C}(x \circ y)$$

not uniformly distributed

def

A = all linear fctns evaluated at assignment a — $A: \mathbb{F}_2^n \to \mathbb{F}_2$ — $A(x) = \sum a_i x_i = a^T \cdot x$

B = all deg 2 fctns evaluated at a — $B: \mathbb{F}_2^{n^2} \to \mathbb{F}_2$ — $B(y) = \sum_{i,j} a_i a_j y_{ij} = (a \circ a)^T \cdot y$

C = all deg 3 fctns evaluated at a — $C: \mathbb{F}_2^{n^3} \to \mathbb{F}_2$ — $C(y) = \sum_{ijk} a_i a_j a_k z_{ijk} = (a \circ a \circ a)^T \cdot z$

Proof contains:

Complete description of truth tables of $\tilde{A}, \tilde{B}, \tilde{C}$ for all inputs x, y, z

only need value at $x = \alpha$, $y = \beta$, $z = \gamma$ but extra info helps us check consistency

---

Test $\quad sc\text{-}\tilde{A}(x_1) \cdot sc\text{-}\tilde{A}(x_2) = \left[\sum a_i x_{1i} \circ \sum a_j x_{2j} = \sum_{i,j} a_i a_j x_{1i} x_{2j} = \sum_{ij} b_{ij} x_{1i} x_{2j}\right]$

picked randomly

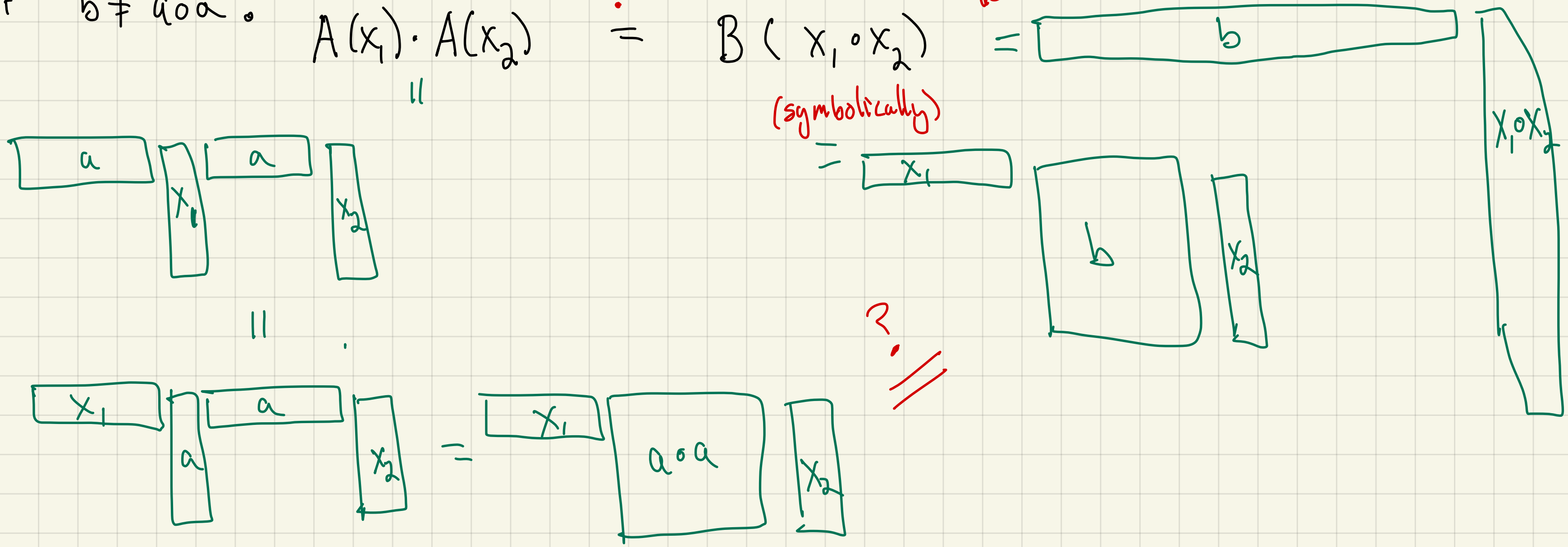
$= sc\text{-}\tilde{B}(x_1 \circ x_2)$

---

if $b = a \circ a$ test passes $\leftarrow$ since "green" equalities hold

if $b \neq a \circ a$: $\quad A(x_1) \cdot A(x_2) \overset{?}{=} B(x_1 \circ x_2) \overset{\text{def}}{=}$

(symbolically)



if $b \neq a \circ a$:

What is prob

$$\boxed{x_1}\ \boxed{a \circ a}\ \boxed{x_2} = \boxed{x_1}\ \boxed{b}\ \boxed{x_2} \ ?$$

Fact: if $\bar{a} \neq \bar{b}$ then $\Pr_{\bar{r} \in_R \{0,1\}^n}[\bar{a} \cdot \bar{r} \neq \bar{b} \cdot \bar{r}] \geq \frac{1}{2}$

if $A \cdot B \neq C$ then $\Pr_{\bar{r}}[A \cdot B \cdot \bar{r} \neq C \cdot \bar{r}] \geq 1/2$

$(a \circ a) \cdot x_2 \stackrel{?}{=} b \cdot x_2$

Yes $\frac{1}{2}$ — Pass with prob 1

No $\frac{1}{2}$ — Pass with prob ? $\frac{1}{2}$

Fact $\Rightarrow$ $\Pr_{x_2}[(a \circ a) \cdot x_2 \neq b \cdot x_2] = 1/2$

if $(a \circ a) \cdot x_2 \neq b \cdot x_2$

then Fact $\Rightarrow$ $\Pr_{x_1}[x_1 (a \circ a) \cdot x_2 \neq x_1 \cdot b \cdot x_2] = 1/2$

$\Rightarrow$ $\Pr[\text{fail test}] \geq 1/4$

So pass test

$\Rightarrow$ safe to assume $b = a \circ a$

Similarly for other test

$c = a \circ a \circ a$

Test $sc\text{-}\tilde{A}(x_1) \cdot sc\text{-}\tilde{A}(x_2) = \left[\sum a_i x_{1i} \circ \sum a_j x_{2j} = \sum_{i,j} a_i a_j x_{1i} x_{2j} = \sum_{ij} b_{ij} x_{1i} x_{2j}\right]$

$= sc\text{-}\tilde{B}(x_1 \circ x_2)$

picked randomly

def $\quad x \circ y = (x_1y_1, x_1y_2, x_1y_3, \dots, x_iy_j, \dots, x_ny_n)$

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B = all deg 2 fctns evaluated at a $\quad B : \mathbb{F}_2^{n^2} \to \mathbb{F}_2 \quad B(y) = \sum_{i,j} a_i a_j y_{ij} = (a \circ a)^T \cdot y$

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$= sc\text{-}\tilde{B}(x_1 \circ x_2)$ (*)

$sc\text{-}\tilde{A}(x) \cdot sc\text{-}\tilde{B}(y) = \left[\sum a_i x_i \circ \sum_{jk} b_{jk} y_{jk} = \sum_{ijk} a_i b_{jk} x_i y_{jk} = \sum a_i a_j a_k x_i y_{jk}\right]$

$= sc\text{-}\tilde{C}(x \circ y)$ (*)

(*) = not uniformly distributed

$$x \circ y = (x_1 y_1, x_1 y_2, x_4 y_3, \ldots, x_i y_j, \ldots, x_n y_n)$$

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- Test $\tilde{A}, \tilde{B}, \tilde{C}$ are all $1/8$-close to linear (i.e. if all linear, PASS; if any one is $1/8$-far FAIL) in $O(1)$ queries

- From now on, use self corrector to get

$$sc\text{-}\tilde{A}, \quad sc\text{-}\tilde{B}, \quad sc\text{-}\tilde{C} \quad \text{for all inputs}$$

$$\Updownarrow \qquad \Updownarrow \qquad \Updownarrow$$

$$a \qquad b = a \circ a? \qquad c = a \circ a \circ a?$$

use β = prob of getting wrong answer in SC that is so small ($\leq \frac{1}{\text{big enough constant}}$) that union bnd over all queries to $sc\text{-}\tilde{A}$, $sc\text{-}\tilde{B}$, $sc\text{-}\tilde{C}$ $\Rightarrow$ unlikely to see error

$x \circ y = (x_1 y_1, x_1 y_2, x_4 y_3, \dots, x_i y_j, \dots, x_n y_n)$

def

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$B : \mathbb{F}_2^{n^2} \to \mathbb{F}_2$ $\quad B(y) = \sum_{i,j} a_i a_j y_{ij} = (a \circ a)^T \cdot y$

C = all deg 3 fctns evaluated at a

$C : \mathbb{F}_2^{n^3} \to \mathbb{F}_2$ $\quad C(y) = \sum_{ijk} a_i a_j a_k z_{ijk} = (a \circ a \circ a)^T \cdot z$

Proof contains:

HUGE

Complete description of truth tables of $\tilde{A}, \tilde{B}, \tilde{C}$ for all inputs x, y, z

only need value at $x = \alpha$, $y = \beta$, $z = \gamma$ but extra info helps us check consistency

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What does verifier need to check in proof?

(1) $\tilde{A}, \tilde{B}, \tilde{C}$ in right form

- all are linear fctns

Can only test ε-close to linear but can self-correct to access the linear fctns.

- correspond to same assignment a

i.e. $\tilde{A}(x) = a^T \cdot x \Rightarrow \tilde{B}(y) = (a \circ a)^T \cdot y \Rightarrow \tilde{C}(z) = (a \circ a \circ a)^T \cdot z$

Test consistency of self corrections

(2) a is satisfying assignment

- all $\hat{C}_i$'s evaluate to 0 on a

(recall $C(a) = (\hat{C}_1(a), \hat{C}_2(a), \dots) \overset{?}{=} (0, 0, \dots 0)$)

complement

$$\sum_i r_i \hat{C}_i(a) = \Gamma + \sum_i a_i \alpha_i + \sum_{i,j} a_i a_j \beta_{ij} + \sum_{i,j,k} a_i a_j a_k \gamma_{ijk} \pmod 2$$

How to do (2):

- call self-correctors $\Rightarrow$ recover linear fctns a, $a \circ a$, $a \circ a \circ a$
- a represents assignment, but we don't know it
- a satisfying $\Leftrightarrow$ $\hat{C}(a) \equiv (\hat{C}_1(a), \hat{C}_2(a), \dots) = (0, 0, \dots, 0)$

Satisfiability Test:

Pick $r \in \mathbb{F}_2^n$

Compute Γ, α_i's, β_{ij}'s, γ_{ijk}'s

query proof to get
$$SC\text{-}\tilde{A}(\alpha_1 \dots \alpha_n) = w_0$$
$$SC\text{-}\tilde{B}(\beta_{11} \dots \beta_{nn}) = w_1$$
$$SC\text{-}\tilde{C}(\gamma_{111} \dots \gamma_{nnn}) = w_2$$

Verify $0 = \Gamma + w_0 + w_1 + w_2 \pmod 2$

Why do this?

if $\forall i$ $\hat{C}_i(a) = 0$

$\Rightarrow \Pr[\text{pass}] = 1$

if $\exists i$ s.t. $\hat{C}_i(a) \neq 0$

Fact $\Rightarrow \Pr[\sum_i r_i \hat{C}_i(a) = 1] = \frac{1}{2}$

so after k times,

$\Pr[\text{pass}] \leq 1/2^k$