6.842 Randomness and Computation	October 3, 2017
Lecture 9	
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1 Stationary distribution

Suppose, we have a Markov chain. It begins with the probability distribution over states $\pi^{(0)}$, and at time t, this distribution equals $\pi^{(t)}$. As we have seen in the previous lecture:

$$\pi^{(t)} = \pi^{(0)} P^t$$

Where P is the transition matrix.

Definition 1 Suppose after starting in some initial probability distribution $\pi^{(0)}$, as the time goes to infinity, the probability distribution converges to some π. We call π a stationary distribution.

If the Markov chain is in a stationary distribution, the probability distribution should remain constant as time progresses, and therefore, it should hold that:

$$\pi(y) = \sum_x \pi(x) P(x, y)$$

Theorem 2 *Every ergodic Markov chain has a unique stationary distribution.*

We do not prove this in class. Various textbooks on probability theory contain a proof of this statement.

As a special case, suppose we have an undirected graph with vertices v_1, v_2,...,v_n. It is easy to check that:

$$\pi = \left(\frac{deg(v_1)}{2|E|}, \frac{deg(v_2)}{2|E|}, ..., \frac{deg(v_n)}{2|E|} \right)$$

is a stationary distribution.

This also implies that a d-regular graph has a uniform stationary distribution. This result can be generalized in the following ways:

- A directed graph, for which every vertex has an in-degree and out-degree d has a uniform stationary distribution.
- A doubly stochastic Markov chain also has a uniform stationary distribution.

This is not true for graphs in general. Furthermore, as we mentioned if the graph is k-partite, it will not have a stationary distribution.

2 Characteristic times

2.1 Definitions

Definition 3 The hitting time h_{ij} between two, not necessarily different vertices, is the expected amount time it will take to visit vertex j for the first time after starting at vertex i. For the case $i = j$ we take the first time it gets back after leaving the initial vertex.

Theorem 4 *For any ergodic Markov chain the hitting time $h_{ii} = \frac{1}{\pi(i)}$. Where π is the stationary distribution.*

Again, the proof can be found in probability textbooks.

Definition 5 The commute time C_{ij} between two vertices i and j is the expected amount of time for the Markov chain to start at i, go to j and come back to i.

From linearity of expectation, it follows that $C_{ij} = h_{ij} + h_{ji}$.

Definition 6 The cover time $C_u(G)$ from vertex u is the expected amount of time, after which a Markov chain that starts at u will visit every single vertex in G.

Definition 7 The cover time C_G of the whole graph is the largest of all the cover times from each of the vertices.

2.2 Examples

1. k_n^* is the undirected complete graph with n vertices, which also has self loops from every edge to itself. Cover time problem for this graph is identical to the coupon collector problem: suppose there are n possible coupons and every time we collect a coupon, we receive one at random. In expectation, after receiving how many coupons, will we collect each coupon at least once? The answer to this question is $\Theta(nlog(n))$, and so is the cover time of our graph.

2. The line graph. It has n vertices arranged in a line, and there are edges between all the neighbors in the line. The cover time is $\Theta(n^2)$. Intuitively, roughly, we expect our Markov chain to come back to the starting leftmost vertex n times before it finally reaches the rightmost vertex. See Feller for more details.

3. The lollipop graph can be constructed by taking a $n/2$-vertex line graph end replacing the last vertex in it by a $n/2$-vertex clique. The cover time is $\theta(n^3)$. Again, intuitively it is similar to the line graph, but every time our Markov chain reaches the clique, it gets "stuck" there, so the cover time is by a factor of n worse than that for the line graph. Again, more information can be found in Feller.

2.3 Upper bound on the cover time.

Theorem 8 *The cover time of a graph G is less than or equal to $O(mn)$. Where, as always, m is the number of edges and n is the number of vertices.*

Proof First of all, let's add self loops to each vertex, so that the Markov chain has a 1/2 probability of staying at vertex v after making a transition from it. For that, we add to a vertex v a number of self-loops equal to the number of edges connected to v.

Now, our Markov chain has to be ergodic, since it is aperiodic and we assume it was irreducible in the first place. Furthermore, the cover time will increase at most twice, because in expectation only one half of all of the time steps will be spent on the self-loops.

Lemma 9 *Suppose there is an edge between vertices i and j. Then the commute time between the vertices i and j: C_{ij} is $O(m)$.*

Proof First of all, suppose we are given that the Markov chain has just completed a transition from i to j. Let t be the amount of time, it spends in expectation to complete a transition from i to j again. Since it is a Markov process, its behavior does not depend on the the states it was in the past, so if it starts at j, it will in expectation take exactly as much time to go to i and to j again, as it will if we are also given that before it started at j it came there from i. Therefore $C_{ij} \leq t$.

Now, lets construct the line graph:

Definition 10 The *line graph* is the graph G' for which:

- The set of vertices V' is the set of all transitions allowed in the original graph, e.i. if there is an edge between vertices i and j, the transitions (i, j) and (j, i) will be in the line graph. We will have only one transition for each of the self-loops.
- The set of edges E' will contain all the pairs of transitions that can happen consecutively, e.i. between (i, j) and (j, k) there will be an edge.

The line graph has a very useful property: it is doubly stochastic. To show this, let's calculate the column sum. If $(u, v)(v, w) \in E'$, we have $Q_{(u,v)(v,w)} = \frac{1}{d_v}$. The column sum is equal to:

$$\sum_{(u,v):(u,v)(v,w)\in E'} Q_{(u,v)(v,w)} = \sum_{(u,v)\in E} \frac{1}{d_v} = 1$$

Which means that this Markov chain is indeed doubly stochastic. Therefore, it has a uniform stationary distribution $\pi(u) = \frac{1}{4m}$. Which means that the hitting time $h_u u = \frac{1}{\pi(u)} = \Theta(m)$.

Therefore, if we start by making a transition (u, v) in expectation we need only $\Theta(m)$ steps to see (u, v) again, which, by the argument we made before is an upper bound on C_{uv}. Therefore $C_{ij} = O(m)$. ■

Now, let's start at a vertex v_0. Because of the connectivity, there has to exist a a spanning tree T that is rooted at v_0.

Let's complete a depth-first traversal of the graph, so we keep track of each vertex twice: once when we firs explore it and the second time when we backtrack from it. We will have a sequence of vertices:

$$v_0, v_1, ..., v_{2n-2}$$

We have

$$C(G) \leq \sum_{j=0}^{2n-3} h_{v_j v_{j+1}}$$

We know that $C_{ij} = h_{ij} + h_{ji}$, so

$$C(G) \leq \sum_{(u,v)\in T} C_{uv} = O\left(\sum_{(u,v)\in T} m\right) = O(mn)$$

.

This proves our theorem.

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