Today:

Undirected S-T Conn revisited

(deterministic logspace)

Undirected s-t connectivity revisited

given: undir G
nodes s,t

Question: are s,t in same component?

an easy case:

def: (N, D, λ) - graph

#nodes, degree, upper bnd on λ_2 of transition matrix

a well-known-fact: Tanner, Alon-Milman

$\forall \lambda < 1, \exists \varepsilon > 0$ s.t. $\forall (N, D, \lambda)$ - graphs G

$+ \forall S$ s.t. $|S| < \frac{N}{2}$

$|\underbrace{N(S)}_{\text{includes } S}| \geq (1+\varepsilon)|S|$ } i.e. G "expands"

fact implies another easy fact: such a G also has $O(\log N)$ diameter

Idea for algorithm in which each component of graph is (N, D, λ) for $\lambda < 1$ + const D (or just $\log n$ - diameter)

- enumerate all D^{ℓ} paths (for $\ell = O(\log N)$) starting at s
- if ever see t, output "connected"

Space requirements:

assume lexicographic ordering on paths (comes from ordering on outedges)

just keep track of DFS path from s:

- const # bits per step of path
- $O(\log n)$ length

Total: $O(\log n)$ bits

$(2, 1, 3, \ldots)$

Behavior:

if s,t not connected, never answers connected

if s,t connected - will find path

Problem: not all graphs are
- (N, D, λ)-graphs for $\lambda < 1$
- or even just $O(\log n)$ diameter
- or even just constant degree

In general, we know:

Thm $\forall$ connected, non-bipartite graphs, $\lambda(G) \leq 1 - \frac{1}{DN^2}$

~~not~~ too good!

What about powering?

if G is (N, D, λ) then G^t is (N, D^t, λ^t)

good or bad?

⊕ reduce 2^{nd} e-val

⊖ increase degree

Need operation which reduces degree w/o killing 2^{nd} e-val

i.e. 1) if it was expander, should remain so
but reduce degree

2) don't need to increase expansion, powering
will do that

Lets start with a "base graph"

Thm 1 $\exists$ const D_e + $((D_e)^{16}, D_e, \frac{1}{2})$-graph

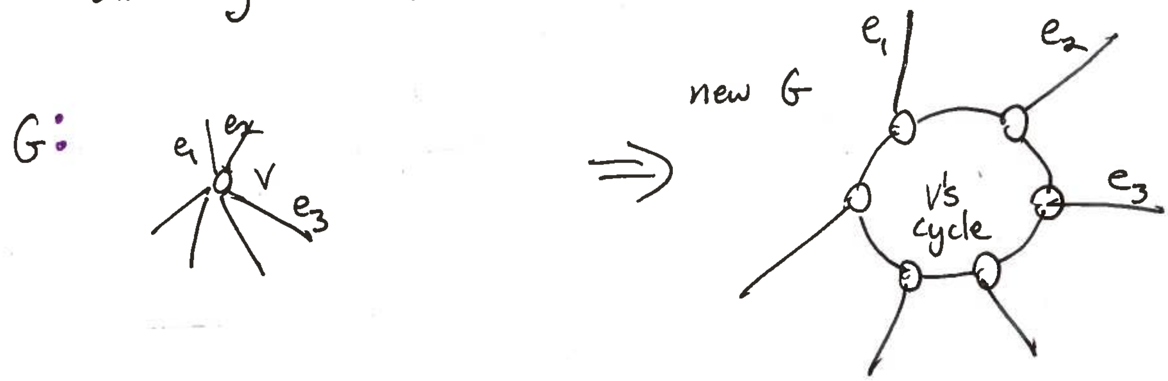
$\uparrow N$ $\uparrow D$ $\uparrow \lambda$

Constant size - can find
this by enumeration

A first issue to handle:

nice to have regular graph of const degree d with same connected components

one way to transform G:



- quadratic blowup in # of nodes, but just once
- can add self loops to deg $< d$ nodes

in both cases, easy to fix neighbor fctn in log space

Could be bad for λ but we'll fix later...

A second issue: representing graphs.

adjacency lists?

Rotation map $Rot_G : [N] \times [D] \to [N] \times [D]$

$Rot_G(v,i) = (w,j)$ if i^{th} edge of v leads to w + j^{th} edge of w leads to v

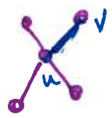
allows back + forth on same edge!

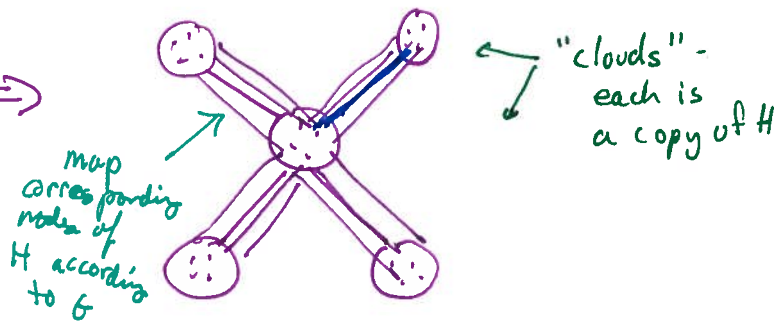
① Replacement Product $G \circledR H$

Given G D-regular N nodes, H d-regular D nodes $\}$ G' with $N \cdot D$ nodes, degree $d+1$ <<

nodes: $v \in G$ replaced by copy of H

edges: each vertex in H_v connected to nbrs in H_v
if u is ith nbr of v in G & v is u's jth nbr
add edge from ith node of H_v to jth node of H_u





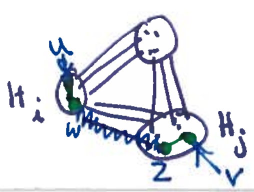
② Zig-zag product $G \circledz H$

Given G D-regular, N nodes, H d-regular D nodes $\}$ G'' with $N \cdot D$ nodes, degree d^2

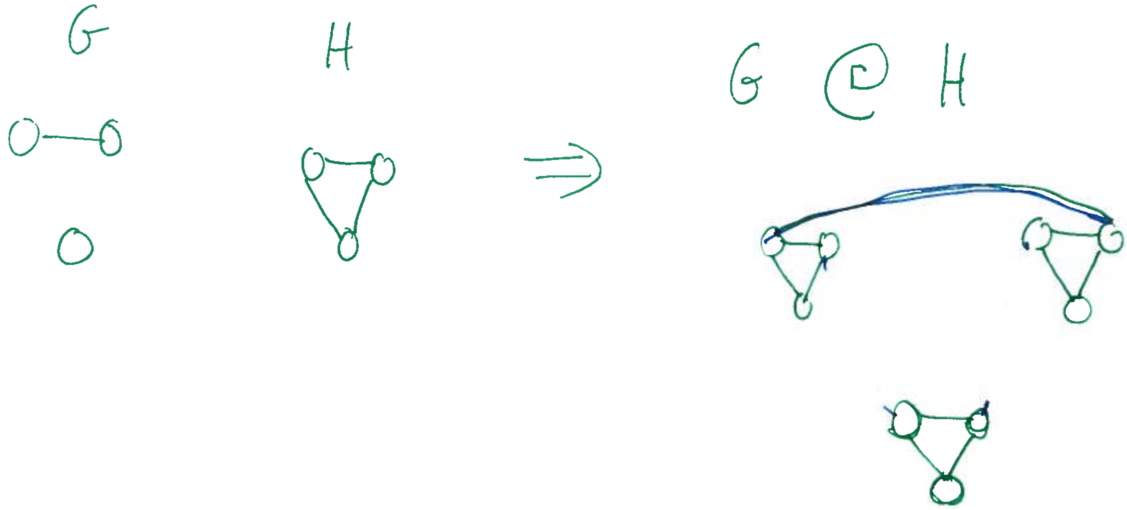
nodes: as in G', $v \in G$ replaced by copy of H

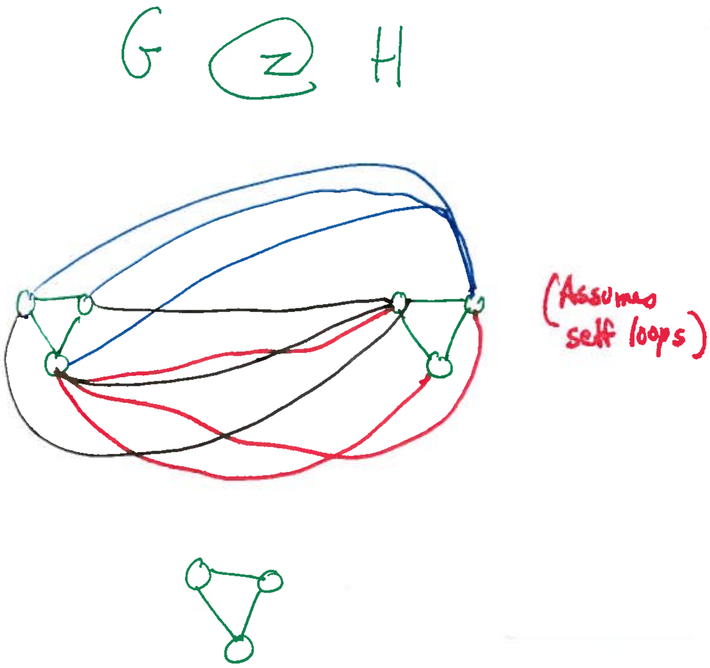
edges: path of length 3 in G' i.e. $(u,v) \in G''$ iff $u \in H_i$ ("cloud i")
$(u,w) \in E(H_i)$, $(w,z) \in E(G)$, $(z,v) \in E(H_j)$ s.t. $z \in H_j$

$\uparrow$ d choices (local) $\quad$ $\uparrow$ 1 choice (between clouds) $\quad$ $\uparrow$ d choices (local) $\quad$ $\}$ degree d^2



Example





Some Intuition:

in terms of cuts –

to find min cut, would want to
break s.t. clouds intact (since clouds are expanders)
$\Rightarrow$ relative cut size should be similar to G's

in terms of random walks –

two extreme cases:

1) distribution far from uniform in each cloud:
then walks on H make it more uniform
+ G step won't affect

2) uniform within clouds but different weights
on clouds:
then walks on H won't affect,
+ walking on G improves slowly

<u>Thm</u> For $\alpha \le 1/2$ For G an (N,D,λ)-graph + H a (D,d,α)-graph, $G \textcircled{z} H$ is $(ND, d^2, \lambda_{G \textcircled{z} H})$-graph

s.t. $\frac{1}{2}(1-\alpha^2)(1-\lambda) \le 1-\lambda_{G \textcircled{z} H}$

So,

$$\lambda_{G \textcircled{z} H} \le 1 - \frac{1}{2}(\underbrace{1-\overset{\le 1/2}{\alpha^2}}_{\ge 3/4})(1-\lambda)$$

$$\le 1 - 3/8(1-\lambda)$$

$$\le 1 - \frac{1}{3}(1-\lambda)$$

$$\le 2/3 + \lambda/3$$ ← still <1, now, let's power it up a few times!

How do we use this?

Main Transformation:

Given: G D^{16}-regular on N nodes

H D-regular on D^{16} nodes (Thm 1)

Transformation:

$l \leftarrow$ smallest int s.t. $\left(1 - \frac{1}{DN^2}\right)^{2^l} < 1/2$

$G_0 \leftarrow G$

$G_i \leftarrow (G_{i-1} \textcircled{z} H)^8$ ← powering

degree reduction (↑ $\textcircled{z}$)

Output: G_l

What are properties of G_ℓ ?

nodes $= N \cdot (D^{16})^\ell$ which is poly(N) since
$\ell = O(\log N)$
+ $D = O(1)$

Lemma if $\lambda(H) \le 1/2$ + G connected, not bipartite
then $\lambda(G_\ell) \le 1/2$

Pf idea $\lambda_{G_0} \le 1 - \frac{1}{DN^2}$

need Claim $\lambda_{G_i} \le \max\{\lambda(G_{i-1})^2, \frac{1}{2}\}$

if Claim, then for $\ell = \Theta(\log DN)$
have $\lambda(G_\ell) \le \max\{\lambda(G_0)^{2^\ell}, \frac{1}{2}\}$

$\le \max\{(1 - \frac{1}{DN^2})^{c \cdot DN^2}, \frac{1}{2}\}$

$\le 1/2$

Then Prove claim by induction on i. ∎

Final Algorithm

1. preprocess G to make regular, nonbipartite
   with same connected components
   (can do by G ⓡ N-cycle + then add self-loops)
   new graph has N^a nodes - or can use idea on pg 4
   + self loops
2. use zigzag + powering transformation to get G_ℓ
3. run expander algorithm on G_ℓ

A final issue:

how do you perform walks in logspace?

need to show that can compute rotation map of G_ℓ given rot map of G, H

use rotation maps!

gives way of going backwards + forwards on a path

⟹ only need to remember a constant # of bits to choose next step of path