6.842 Randomness and Computation	October 18, 2017
Homework 5	
Lecturer: Ronitt Rubinfeld	*Due Date: October 25, 2017*

The following problems are to be turned in.

1. **(Edge expansion)** An n-vertex d-regular graph $G = (V, E)$ is called an (n, d, ρ)-edge expander if for every subset S of vertices satisfying $|S| \leq n/2$,

$$|E(S, \overline{S})| \geq \rho d |S|$$

where $E(S, T)$ denotes the set of edges $(u, v) \in E$ with $u \in S$ and $v \in T$.

Prove that for every n-vertex d-regular graph, there exists a subset S of $\leq n/2$ vertices such that

$$E(S, \overline{S}) \leq \frac{dn}{4}(1 + \frac{1}{n-1})$$

(Hint: use the probabilistic method). Conclude that there does not exist an (n, d, ρ)-edge expander for any constant $\rho > 1/2$: more formally, for $\rho > 1/2$, there exists n_0 such that for all d and $n > n_0$, there is no (n, d, ρ) edge expander.

2. **(Random bipartite graphs are good vertex expanders)** A graph $G = (V, E)$ is called an (n, d, c)-vertex expander if it has n vertices, the maximum degree of a vertex is d and for every subset $W \subseteq V$ of cardinality $|W| \leq n/2$, the inequality $|N(W)| \geq c|W|$ holds, where $N(W)$ denotes the set of all vertices in $V \setminus W$ adjacent to some vertex in W. By considering a random bipartite 3-regular graph on $2n$ vertices obtained by picking 3 random permutations between the 2 color classes, one can prove that there exists $c > 0$ such that for every n there exists a $(2n, 3, c)$-vertex expander.

For this homework, just prove that for any subset L of size at most $n/2$ of the "left vertices", (with constant probability) there are at least $(1 + \epsilon)|L|$ "right" neighbors.

It is fine to allow multiple edges in the construction.