Edge Matching with Inequalities, Triangles, Unknown Shape, and Two Players

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In an ***edge-matching puzzle***, we are given several tiles (usually identical in shape), where each tile has a label on each edge, and the goal is to place all the tiles into a given shape such that shared edges between adjacent tiles have compatible labels. In ***unsigned*** edge matching, labels are compatible if they are identical; in ***signed*** edge matching, labels have signs, and two labels are compatible if they are negations of each other. Physical edge-matching puzzles date back to the 1890s; perhaps the most famous example is *Eternity II* which offered a US$2,000,000 prize for a solution before 2011.

Previous work. The complexity of edge-matching puzzles has been studied since 1966. The most relevant work to this paper is from two past JCDCG conferences. In 2007, Demaine and Demaine [DD07] proved that signed and unsigned edge-matching square-tile puzzles are NP-complete and equivalent to both jigsaw puzzles and polyomino packing puzzles. In 2016, Bosboom et al. [BDD+17] proved that signed and unsigned edge-matching square-tile puzzles are NP-complete even when the target shape is a $1 \times n$ rectangle, and furthermore hard to approximate within some constant factor.

Our results. In this paper, we analyze the complexity of several variants of the edge-matching problem by varying the tile shape, target board shape, label compatibility condition, and number of players. Table 1 summarizes our results, which we now describe.

Compatibility	Board	Tiles	Players	Complexity
$<$	$1 \times n$	square	1-player	NP-complete
$\leq$	$m \times n$	square	1-player	P
Signed/unsigned	$1 \times n$	square	1-player	(2-)ASP-hard,* #P-complete
Signed/unsigned	$1 \times n$	equilateral triangle	1-player	NP-complete, #P-complete
Signed/unsigned	$1 \times n$	right triangle (hypotenuse contact)	1-player	NP-complete, #P-complete
Signed/unsigned	$\frac{\sqrt{2}}{2} \times n$	right triangle (leg contact)	1-player	P, #P-complete
Signed/unsigned	$O(1) \times n$	square/triangular with $O(1)$ colors	1-player	P
Signed/unsigned	shapeless	square	1-player	ASP-hard, #P-complete
Signed/unsigned	$1 \times n$	square	impartial 2-player	PSPACE-complete
Signed	$1 \times n$	square	partisan 2-player	PSPACE-complete

Table 1: Our results on edge-matching puzzles. *Our proof gives ASP-hardness for $1 \times n$ edge matching only when at least one boundary edge is colored; otherwise, each solution can be rotated 180 degrees to form another valid solution, so we get 2-ASP-hardness (NP-hard to find a third solution given two).

Inequality edge matching. Our most involved result is an NP-hardness proof for a new "$<$" compatibility condition, where edge labels are numbers, horizontally adjacent edges match if the left edge's number is less than the right edge's number, and vertically adjacent edges match if the top edge's number is less than the bottom edge's number. We prove NP-hardness of $<$-compatible $1 \times n$ edge matching by reduction from another new NP-hard problem, interval-pair cover. The $\leq$-compatibility condition (where equal numbers also match, or we assume all labels are distinct) turns out to be substantially easier: even rectangular puzzles turn out to be always solvable, and we give a polynomial-time algorithm.

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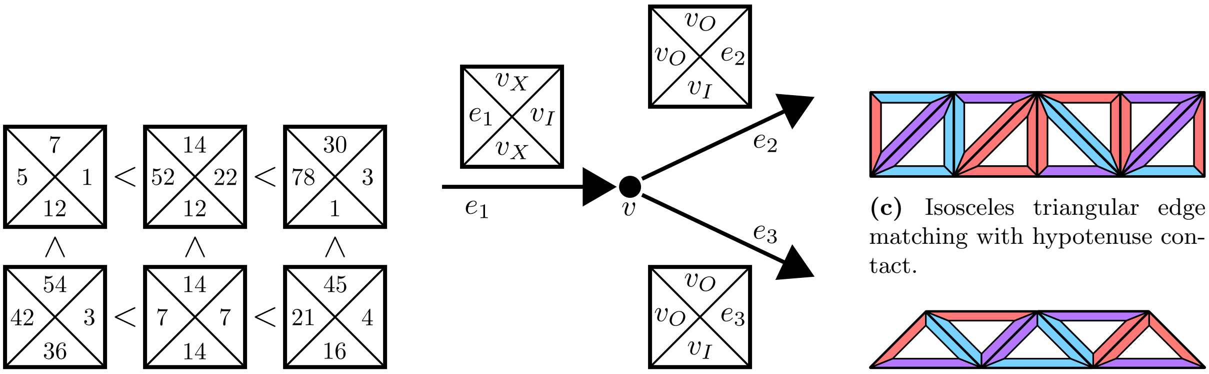

(a) A solution to a 2×3 $<$-compatible edge-matching puzzle.

(b) A gadget from the ASP-hardness proof for $1 \times n$ edge matching.

(c) Isosceles triangular edge matching with hypotenuse contact.

(d) Isosceles triangular edge matching with leg contact.

ASP/#P-completeness for $1 \times n$ edge matching. Of independent interest, we prove ASP-completeness of Hamiltonian path with specified start and end vertices in max-degree-3 *directed* graphs, by modifying the clause gadget from Plesník's NP-hardness proof [Ple79] and parsimoniously reducing from 1-in-3SAT instead of 3SAT. We then use this result to prove ASP-completeness for signed and unsigned $1 \times n$ edge-matching puzzles when the left boundary edge is colored (to prevent trivial rotation of solutions), and 2-ASP-hardness and #P-completeness even if the boundary is colorless.

Triangular edge matching. The conclusion of [BDD+17] claimed that the paper's results extended to equilateral-triangle edge matching, but the proposed simulation of squares by triangles is incorrect because it constrains the orientation of the simulated squares. We extend our $1 \times n$ parsimonious proof to obtain NP-hardness (but not ASP-hardness) for signed and unsigned edge matching with equilateral triangles, with or without reflection.

For right isosceles triangles, there are two natural "$1 \times n$" arrangements. For clarity, we assume the legs of the triangles have length 1. If we still want a height-1 tiling, then length-$\sqrt{2}$ hypotenuses are forced to match, so matching is NP-complete by simulation of squares. But if we ask for a height-$\frac{\sqrt{2}}{2}$ tiling, so only legs match, we show surprisingly that both signed and unsigned edge matching can be solved in polynomial time using an algorithm based on Eulerian paths.

Of independent interest, we characterize when a directed graph admits an Eulerian "path" that alternates between going forward and going backward along edges.

Shapeless edge matching. We prove that square-tile edge-matching puzzles remain NP-, ASP-, and #P-complete when the goal is to connect all tiles in any (unspecified) single connected shape, with either signed or unsigned compatibility. The proof builds a unique spiral frame that effectively forces a $1 \times n$ edge-matching puzzle with a fixed left boundary color.

2-player edge matching. We consider natural 2-player variants of $1 \times n$ edge-matching puzzles, where the players alternate placing a tile in the leftmost empty cell and the first player unable to move loses (normal play). We prove PSPACE-completeness for both signed and unsigned square-tile edge matching when players can play any remaining tile, and for signed edge matching when players play from separate pools of tiles.

References

[BDD+17] Jeffrey Bosboom, Erik D. Demaine, Martin L. Demaine, Adam Hesterberg, Pasin Manurangsi, and Anak Yodpinyanee. Even $1 \times n$ edge-matching and jigsaw puzzles are really hard. *Journal of Information Processing*, 25:682–694, 2017.

[DD07] Erik D. Demaine and Martin L. Demaine. Jigsaw puzzles, edge matching, and polyomino packing: Connections and complexity. *Graphs and Combinatorics*, 23(1):195–208, 2007.

[Ple79] Ján Plesník. The NP-completeness of the Hamiltonian cycle problem in planar diagraphs with degree bound two. *Information Processing Letters*, 8(4):199–201, April 1979.