Folding Polyominoes with Holes into a Cube*

Oswin Aichholzer[a], Hugo A. Akitaya[b], Kenneth C. Cheung[c], Erik D. Demaine[d], Martin L. Demaine[d], Sándor P. Fekete[e], Linda Kleist[e], Irina Kostitsyna[f], Maarten Löffler[g], Zuzana Masárová[h], Klara Mundilova[d], Christiane Schmidt[i]

[a] *Institute for Software Technology, Graz University of Technology,* oaich@ist.tugraz.at
[b] *School of Computer Science, Carleton University,* hugoakitaya@gmail.com
[c] *NASA Ames Research Center,* kenny@nasa.gov
[d] *CSAIL, Massachusetts Institute of Technology,* {edemaine,mdemaine,kmundil}@mit.edu
[e] *Department of Computer Science, TU Braunschweig,* {s.fekete,l.kleist}@tu-bs.de
[f] *Mathematics and Computer Science Department, TU Eindhoven,* i.kostitsyna@tue.nl
[g] *Department of Information and Computing Science, Universiteit Utrecht,* m.loffler@uu.nl
[h] *IST Austria, Klosterneuburg,* zuzana.masarova@ist.ac.at
[i] *Department of Science and Technology, Linköping University,* christiane.schmidt@liu.se

Abstract

When can a polyomino piece of paper be folded into a unit cube? Prior work studied tree-like polyominoes, but polyominoes with holes remain an intriguing open problem. We present sufficient conditions for a polyomino with one or several holes to fold into a cube, and conditions under which cube folding is impossible. In particular, we show that all but five special "basic" holes guarantee foldability.

Keywords: folding, origami folding, cube, polyomino, polyomino with holes, non-simple polyomino

1. Introduction

Given a piece of paper in the shape of a polyomino, i.e., a polygon in the plane formed by unit squares on the square lattice that are connected edge-to-edge, does it have a folded state in the shape of a unit cube? The standard rules of origami apply [3]; in particular, we allow each unit-square face to be covered by multiple layers of paper. Examples of this decision problem are given by the three puzzles

*A preliminary extended abstract appeared in the Proceedings of the 31st Canadian Conference on Computational Geometry [1].

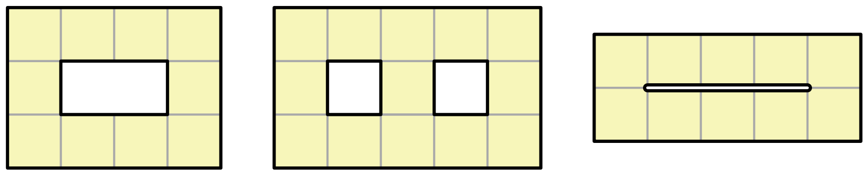

Figure 1: Three polyominoes that fold along grid lines into a unit cube, from puzzles by Nikolai Beluhov [2].

by Nikolai Beluhov [2] shown in Figure 1. We encourage the reader to print out the puzzles and try folding them.

Prior work [4] studied this decision problem extensively, introducing and analyzing several different models of folding. Beluhov [2] implicitly defined a ***grid model*** with the puzzles in Figure 1: Fold only along grid lines of the polyomino; allow only orthogonal fold angles[1] ($\pm 90°$ and $\pm 180°$); and forbid folding material strictly interior to the cube. In this model, the prior work [4] characterizes which ***tree-shaped*** polyominoes (whose unit squares are connected edge-to-edge to form a tree dual graph) lying within a $3 \times n$ strip can fold into a unit cube, and exhaustively characterizes which tree-shaped polyominoes of ≤ 14 unit squares fold into a unit cube.

Notably, however, the polyominoes in Figure 1 are not tree-shaped, and their interior is not even simply connected: The first puzzle has a hole, the second puzzle has two holes, and the third puzzle has a degenerate (zero-area) hole or ***slit***. Arguably, these holes are what makes the puzzles fun and challenging. Therefore, in this paper, we embark on characterizing which polyominoes with hole(s) fold into a unit cube in the grid model. Although we do not obtain a complete characterization, we give many interesting conditions under which a polyomino does or does not fold into a unit cube.

The problem is sensitive to the choice of model. The other main model that has been studied in past work is the more flexible ***half-grid model***, which allows orthogonal and diagonal folds between half-integral points, as in Figure 2. The prior work [4] shows that *all* polyominoes of at least ten unit squares can fold into a unit cube in the half-grid model, leaving only a constant number of cases to explore, which were tackled recently [5]. Therefore, we focus on the grid model, which matches the puzzles of Beluhov [2].

[1]The fold angle of a fold measures the deviation from the flat (unfolded) state, i.e., 180° minus the dihedral angle between the two incident faces.

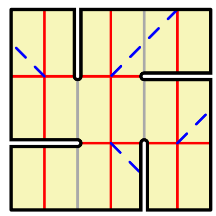

Figure 2: One way to fold a cube in the half-grid model, adapted from [4, Fig. 5(b)]. In all our figures, solid/red lines denote mountain folds, dashed/blue lines denote valley folds, light/grey lines denote grid lines, and bold/black lines denote the polyomino boundary.

If we generalize the target shape from a unit cube to polycube(s), there are polyominoes that fold in the grid model into all polycubes of at most a given surface area [6]. If we further forbid overlapping unit squares (polyhedron unfolding/nets instead of origami), this fold-all-polycubes problem has been studied for small polycubes [7], and there is extensive work on finding polyominoes that fold into multiple (two or three or more) different boxes [8, 9, 10, 11, 12].

Our Results

1. We show that any hole that is not one of five ***basic*** shapes of holes (see Figure 3) always guarantee that a polyomino containing the hole folds into a cube; see Theorem 1 in Section 3.1. Polyominoes with exactly one of the five basic holes only sometimes allow folding into a cube.
2. We identify combinations of two (of the five basic) holes that allow the polyomino to fold into a cube; see Section 3.2.
3. We show that certain of the five basic holes or their combinations do not allow folding into a cube, that is, we show that subclasses of polyominoes with only specific basic hole(s) cannot be folded into a unit cube; see Section 4.
4. We present an algorithm that checks a necessary local condition for folding into a cube; see Section 4.3.
5. Whether this condition also constitutes a sufficient condition remains an open question; see Section 5.
6. We conjecture that a slit of size 1 (see Figure 3, second from left) never affects whether a polyomino can fold into a cube; see Section 4.2. However, we show that a slit of size 1 can be the deciding factor for foldability for larger polycubes.

2. Notation

A ***polyomino*** is a connected polygon P in the plane formed by joining together $|P| = n$ unit squares on the square lattice. We refer to the vertices of the n unit squares forming P as the ***grid points*** of P. We view P as an ***open*** region (excluding its boundary) which includes the n open unit squares of the form $(x_i, x_i+1)\times(y_i, y_i+1)$ as well as *some* of the shared unit-length edges (and grid points) among these n unit squares. Notably, we do not require P to include the common edge between every adjacent pair of squares; if such an edge is missing from P, we call the edge a ***slit edge***. But there must be at least $n-1$ unit-length edges in P so that P is (interior-)connected.

A ***hole*** of a polyomino P is a bounded connected component of P's exterior, whose boundary is one of the connected components of P's boundary other than the outermost one. We assume that P has no holes that are just a single grid point, because such holes do not affect foldability, so we can fill them in (add them to P). We call a hole a ***slit*** if it has zero area (and is more than a single point), and thus consists entirely of one or more slit edges. We call a hole ***basic*** if it has one of the following shapes (refer to see Figure 3):

1. A unit square
2. A slit of size 1 (a single slit edge)
3. A slit of size 2 (L-shaped or straight)
4. A U-slit of size 3

A ***unit cube*** $\mathcal{C}$ is a three-dimensional polyhedron with six unit-square faces and volume of 1. We refer to the vertices of $\mathcal{C}$ as ***corners***.

In this paper, we study the problem of folding a given polyomino P with holes to form $\mathcal{C}$, allowing creases along edges of the lattice with fold angles of $\pm 90°$ or $\pm 180°$. In all our figures, solid/red lines denote mountain folds, dashed/blue lines denote valley folds, light/grey lines denote grid lines, bold/black lines denote the polyomino boundary, bold dotted/purple lines denote creases folded by $\pm 180°$, and thin dotted/purple lines denoted creases folded by $\pm 90°$ (the purple/dotted creases could be mountain or valley).

Some crease patterns give numbers on the unit squares to indicate which face they fold onto in a "real-world" six-sided die, where opposite faces sum up to 7.

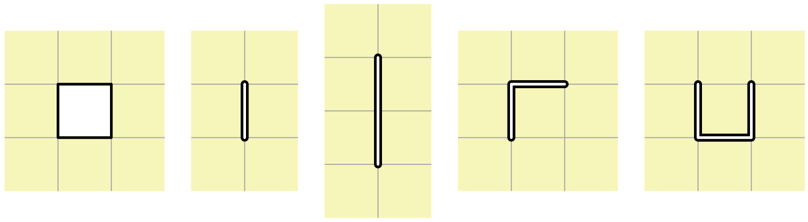

Figure 3: The five basic holes: a unit square, a slit of size 1, a straight slit of size 2, an L-slit of size 2, and a U-slit of size 3.

3. Polyominoes That Do Fold

In this section, we present polyominoes that fold. We start with polyominoes that contain a hole guaranteeing foldability.

3.1. Single-Hole Polyominoes

In this section, we show that any hole different from a basic hole guarantees foldability.

Theorem 1. *If a polyomino P contains a hole h that is not basic, then P folds into a cube.*

Proof. By the definition of basic hole, because h is non-basic, it must be a superset of either two unit squares, a unit square and a unit slit, or a slit of size 3 that is not a U-slit. In the case it contains two unit squares sharing a grid point, h must be a superset of one of the holes in Figure 5 (a)–(b) up to rotation. If it contains a unit square and a unit slit sharing a grid point, then h is a superset of Figure 5 (e) up to reflection and rotation. Else, h must be a superset of the slits in Figure 5 (c), (d), (f), (g) because those are all possible slits of size 3 that are not U-shaped up to reflection and rotation. Then, we distinguish the cases where h contains

- Two unit squares sharing an edge
- Two unit squares sharing a grid point
- A unit square and an incident slit
- A slit of length at least 3 (straight, zigzagged, L-shaped, or T-shaped)

In a first step, we show that if h contains one of the four above holes, we may assume that it contains exactly this hole. Let h be a hole containing a hole h' of the above type. By definition of a hole, h needs to be enclosed by solid squares. Thus we can sequentially fold the squares of P in columns to the left and right of h' on top of the columns directly left and right of h', respectively, as illustrated in Figure 4. Afterwards, we fold the squares of P in rows to the top and bottom of h'

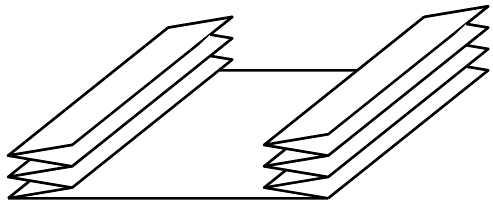

Figure 4: Folding strategy to reduce to seven cases.

on top of the rows directly top and bottom of h', respectively. We call the resulting polyomino P'. Note that because h is a hole of P, all neighboring squares of h' exist in P'. Thus we may assume that we are given one of the seven polyominoes depicted in Figure 5, where striped squares may or may not be present. Note that we can assume that no additional slits are present.

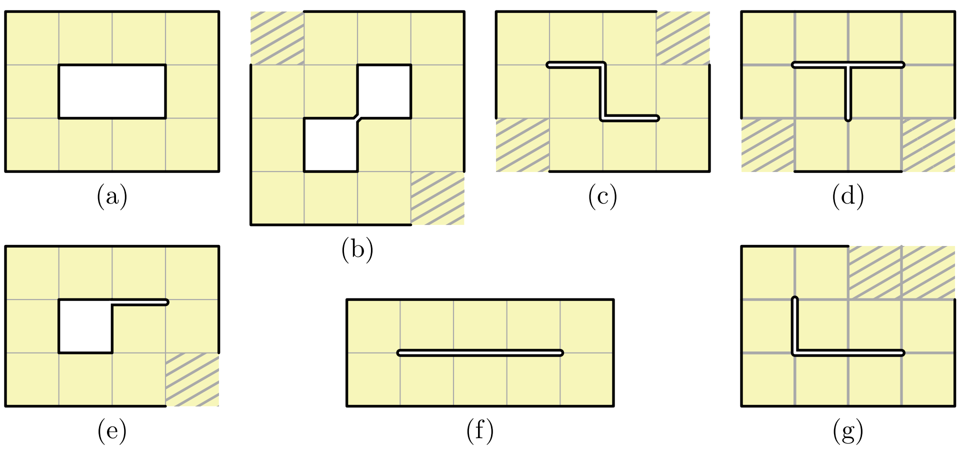

Figure 5: Any polyomino with a hole that is not basic can be reduced to one of the seven illustrated cases; striped squares may or may not be present.

Secondly, we present strategies to fold the polyomino into $\mathcal{C}$. Note that the case if h' consists of two squares sharing only a grid point, we can fold the top row on its neighboring row and obtain the case where h' consist of a square and an incident slit. For an illustration of the folding strategies for the remaining six cases consider Figure 6. □

Are basic holes ever helpful?

In fact, four of the five basic holes sometimes allow foldability, as illustrated in Figure 7. Note that the U-slit of size 3 reduces to the square hole by a ±180°

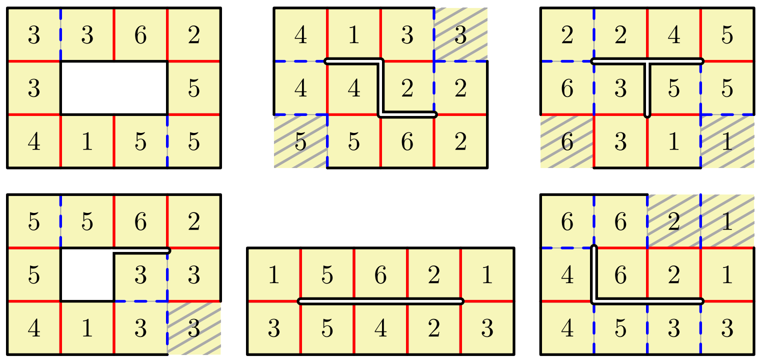

Figure 6: Crease pattern of cube foldings. Mountain folds are shown in solid red, valley folds in dashed blue. Squares with the same number cover the same face of the cube.

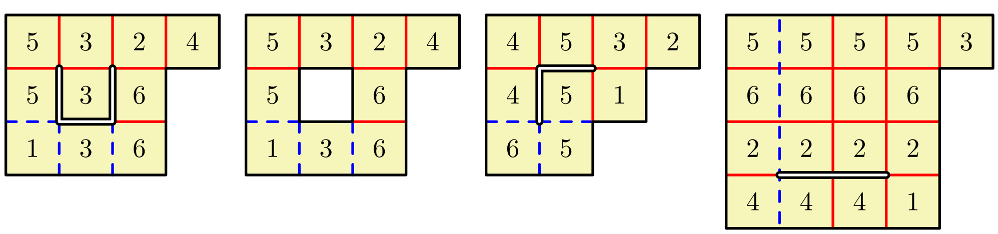

Figure 7: Four basic holes may be helpful. Mountain folds are shown in solid red, valley folds in dashed blue.

mountain or valley fold.

In Theorem 15, we show that the slit of size 1 never helps to fold a rectangular polyomino. Moreover, we show in Lemma 14 that the crease pattern around the slit behaves as if the slit was nonexistent, i.e., the only option to make use of the slit is to push part of the polyomino through the slit. In fact, we conjecture that the slit of size 1 never helps to fold a polyomino into $\mathcal{C}$.

3.2. Combinations of Two Basic Holes

In this section, we consider combinations of two basic holes that fold. For a polyomino with two holes, for which the lowest grid point v of the upper hole has a higher y-coordinate than the upper grid point w of the lower hole, we denote the number of unit-square rows between w and v as the ***number of rows between*** the two holes. Analogously, we define the ***number of columns between*** two holes.

Theorem 2. *A polyomino with two vertical straight size-2 slits with at least two columns and an odd number of rows between them folds.*

Proof. As in the previous section, we first fold all rows between the slits together to one row; this is possible because there is an odd number of rows between the slits. Then, all the surrounding rows and columns are folded towards the slits. Finally, we fold the columns between the slits to reduce their number to two or three. Depending on whether the number of columns between the slits was even or odd, this yields a shape as shown in Figure 8 (a) and (b), respectively, where the striped squares may be (partially) present. In all cases, the two shapes fold as indicated by the illustrated crease pattern. Note that in Figure 8 (b) the polyomino is of course connected, which implies that at least one square of the central column is part of the polyomino, i.e., a square with label 6 is used. □

If we have a polyomino with exactly two slits that have only one or no column between them, then the shape cannot be folded as can be verified by the algorithm of Section 4.3. In the following theorems we call a U-slit which has the open part at the bottom an A-slit. If the orientation of the U-slit is not relevant, then we call it a C-slit.

Theorem 3. *A unit cube can be folded from any polyomino with an A-slit and a unit-square hole/C-slit in the same column above it, having an even number of rows between them.*

Proof. We can assume that the upper hole is a unit square, as the flaps generated by a C-slit can always be folded away. Similar to before we fold away all surrounding rows and columns and reduce the number of rows between the A-slit and the unit-square hole to two. This yields the shape of Figure 8 (c), which can be folded as indicated by the crease pattern. □

Note that if the bottom slit is a U-slit, then the shape of Figure 8 (c) does not fold, again verified by the algorithm of Section 4.3.

Theorem 4. *A polyomino with an A-slit and a unit-square hole/C-slit below it and separated by an odd number of rows, folds, regardless in which columns they are.*

Proof. As before, we assume that the lower hole is a unit square, fold away the surrounding rows and columns, and reduce the number of rows between the two slits/holes to one. Furthermore, we fold the columns between the slits/holes such that at most two columns remain between the two slits/holes. Consequently, we

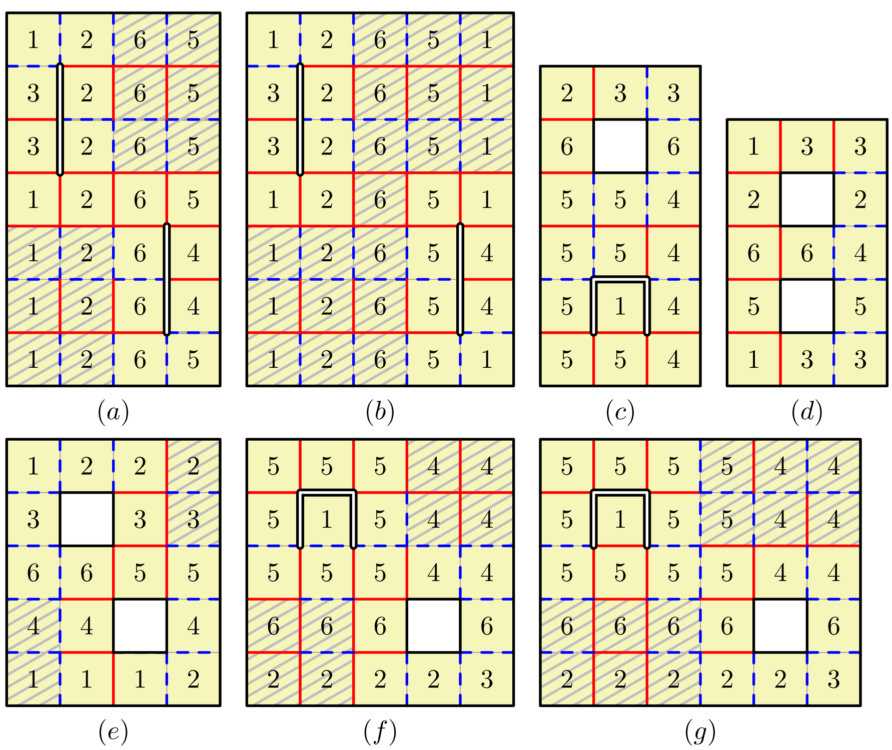

Figure 8: Combinations of two basic holes that are foldable with and without (part of) the striped region. Mountain folds are shown in solid red, valley folds in dashed blue.

obtain one of the shapes shown in Figure 8 (d) to (g). All of them fold, with or without the striped region. Note that the upper unit-square holes in Figure 8 (d) and (e) can be replaced by an A-slit which can be folded away. □

Note that if the two holes are in the same or neighbored column(s) (Figure 8 (d) and (e)), then independently of the orientation of the U-slits or whether they are unit-square holes, any combination folds, yielding Theorem 5. In the other cases, the unit square incident to all three slit edges constitutes the only unit square that covers the face '1' in the unit cube.

Theorem 5. *A polyomino with two unit-square holes which are in the same or in neighboured column(s) and have an odd number of rows between them folds.*

4. Polyominoes That Do Not Fold

In this section, we identify basic holes and combinations of basic holes that do not allow the polyomino to fold. First, we present some results that show how the paper is constrained around an interior grid point v. In particular, we consider situations when the induced polyomino of the four unit squares A, B, C, D incident to v is connected; for an example consider Figure 9.

Lemma 6. *Four unit squares incident to a polyomino grid point v for which the induced polyomino is connected, cannot cover more than three faces of $\mathcal{C}$.*

Proof. The grid point v is incident to four unit squares in P. As grid points of P are mapped to corners of $\mathcal{C}$ and all corners of $\mathcal{C}$ are incident to 3 faces, v is incident to only 3 faces in $\mathcal{C}$. □

Lemma 7. *Four unit squares incident to a grid point v not on the boundary of P cannot cover more than two faces of $\mathcal{C}$. In particular, at least two collinear incident creases are folded by* $\pm 180°$.

Proof. Let A, B, C, and D be the unit squares incident to v in circular order; see the left of Figure 9. By Lemma 6, A, B, C, and D cover at most three faces of $\mathcal{C}$. Hence, at least two unit squares map to the same face of $\mathcal{C}$; these can be edge-adjacent or diagonal.

In the first case, assume without loss of generality that A and B map to the same face. Hence, the crease between them must be folded by $\pm 180°$. Then C and D must also map to the same face of $\mathcal{C}$ to maintain the paper connected. Consequently, the crease between C and D is folded by $\pm 180°$.

In the latter case, let without loss of generality A and C map to the same face of $\mathcal{C}$. As they are both incident to v, only two options of folding those two unit squares on top of each other exist. Either the edge between A and B gets folded on top of the edge between B and C, this leaves a diagonal fold on B, a contradiction, or the edge between A and D gets folded on top of the edge between B and C, which results in D being mapped to C, and those are two adjacent unit squares, by the above argument two collinear incident creases must be folded by $\pm 180°$. □

Figure 9: Illustration of Lemmas 7 and 8. The thick dotted/purple lines represent creases fold by $\pm 180°$; they could be mountain or valley.

Lemma 8. *Consider a grid point v that is not on the boundary of a polyomino P that folds into $\mathcal{C}$. If one crease of v is folded by $\pm 180°$, then the incident collinear crease is also folded by $\pm 180°$.*

Proof. Without loss of generality, we show that if the left horizontal crease of v is folded by $\pm 180°$, the same holds for the right horizontal crease. We denote the left and right adjacent grid points of v by a and b, respectively, as indicated in Figure 9, right.

Suppose for a contradiction, that the right crease is not folded by $\pm 180°$. Then, by Lemma 7, both vertical creases are folded by $\pm 180°$. In particular, a and b are mapped to the same corner of $\mathcal{C}$ and thus the edges av and bv coincide. Hence, because av is folded by $\pm 180°$, bv is also folded by $\pm 180°$. □

Lemmas 7 and 8 imply that:

Corollary 1. *Let $k, n \geq 2$ and let P be a polyomino containing a rectangular $k \times n$-subpolyomino P' whose interior does not contain any boundary of P. Then, in every folding of P into $\mathcal{C}$, all collinear creases of P' are either folded by $\pm 90°$ or by $\pm 180°$. Moreover, either all horizontal or all vertical creases of P' are folded by $\pm 180°$; see Figure 10.*

Proof. First, suppose for a contradiction that there exist two collinear creases, one of which is folded by $\pm 90°$ and the other by $\pm 180°$. Then there also exists an interior grid point of P' where the crease type of the two collinear edges changes from ±90° to ±180°. However, by Lemma 8, if one is folded by ±180°, then both are. A contradiction.

Second, suppose that not all horizontal creases are folded by ±180°. Then, by the first statement, there exists a row in which no grid point has a horizontal edge that is folded by ±180°. By Lemma 7, all vertical creases incident to the grid points of this row are folded by ±180°. Because all collinear edges behave alike, it follows that all vertical creases are folded by ±180°. □

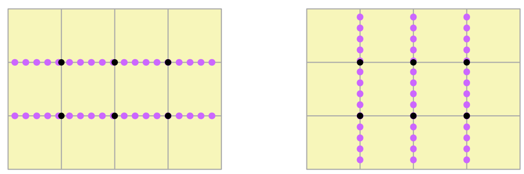

Figure 10: Illustration of Corollary 1.

Corollary 2. *Let P be a rectangular $k \times n$-polyomino without any holes, then P does not fold into $\mathcal{C}$.*

4.1. Polyominoes with Unit Square, L-Shaped, and U-Shaped Holes

We begin by examining the possible foldings of a polyomino containing a unit-square hole. Suppose that a given polyomino P with a unit-square hole h folds into a cube. Furthermore, let the shape of h no longer be a square in the folded state; we say hole h is folded in a ***non-trivial*** way. For an example consider Figures 11 and 12. Then, in the folded state, either all edges of h are mapped to the same edge of $\mathcal{C}$, or two pairs of edges are glued forming an L-shape. In the following, we show that if P folds into $\mathcal{C}$, the first case is impossible, while the second produces a specific crease pattern around h.

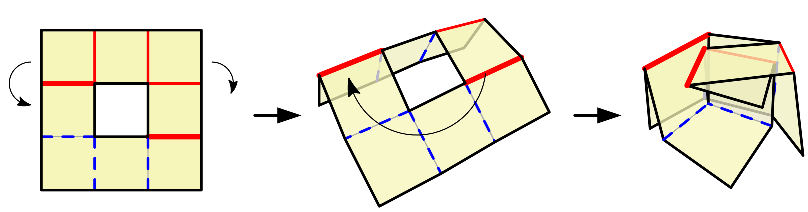

Figure 11: An example of a non-trivial fold of a 3×3 square with a unit square hole. The crease pattern is a special case of the one in Figure 13.

Lemma 9. *The four edges of a unit-square hole h of a polyomino P that folds into $\mathcal{C}$ are not mapped to the same edge of $\mathcal{C}$ in the folded state.*

Proof. We denote the four unit squares of the polyomino edge-adjacent to h by A, B, C, and D, and the four unit squares adjacent to h via a grid point as F_1, F_2, F_3, and F_4, as illustrated in Figure 12. Assume for a contradiction that all edges of h are mapped to the same edge of $\mathcal{C}$. Consider A, F_1, and B in the folded state. As the two corresponding edges of h are glued together, the three faces must be pairwise perpendicular. The similar statement holds for the triples $\{B, F_2, C\}$, $\{C, F_3, D\}$, and $\{D, F_4, A\}$. This results in a configuration as illustrated in the right of Figure 12.

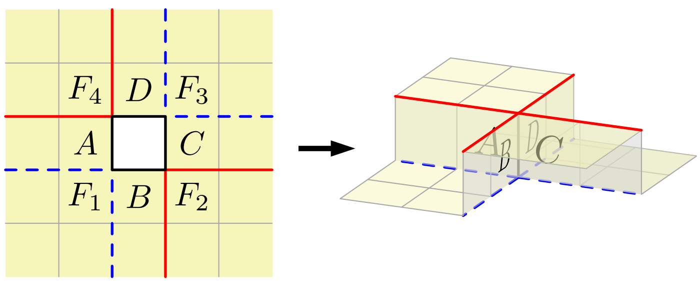

Figure 12: Four edges of a square hole glued together.

Because the faces A, B, C share an edge of $\mathcal{C}$ in the folded state such that A and B, as well as B and C are perpendicular, A and C must cover the same face of $\mathcal{C}$. Likewise, B and D cover the same face of $\mathcal{C}$. If P folds into $\mathcal{C}$, then F_1 and F_3, as well as F_2 and F_4 are mapped to the same faces of $\mathcal{C}$. Suppose, without loss of generality, that in the folded state A lies in a more outer layer than C. Then, F_1 and F_4 are in a more outer layer than F_3 and F_2, respectively. Thus, B connects the more inner layer of F_2 to the more outer layer of F_1, and at the same time D connects the inner layer of F_3 to the outer layer of F_4. Hence, B and D intersect, which is impossible. Therefore, if the polyomino folds into a cube, the four edges of a square hole cannot all be mapped to the same edge of $\mathcal{C}$. □

It follows that the only non-trivial way to glue the edges of a square hole h of a polyomino folded into a cube is to form an L-shape. We use this to show the following fact:

Lemma 10. *Let P be a polyomino with a unit-square hole that folds into $\mathcal{C}$. In every folding of P into $\mathcal{C}$ where h is folded non-trivially (i.e., h is not mapped to a*

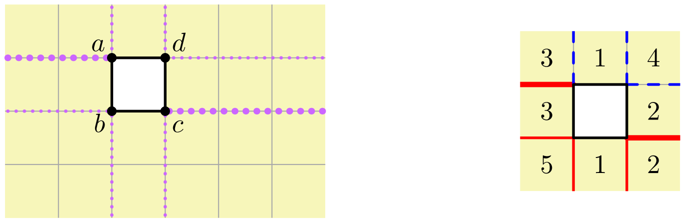

Figure 13: Thinner dotted/purple lines indicate creases folded by $\pm 90°$, while thicker dotted/purple lines indicate creases folded by $\pm 180°$. Left: crease pattern around a unit-square hole folding into an L-shape when grid points a and c are mapped to the same corner of $\mathcal{C}$; creases shown in purple can be both mountain or valley. Right: numbers indicate the face of the cube in the folded state; mountain folds are shown as red solid lines, and valley folds as blue dashed lines.

square), the crease pattern of the unit squares incident to h is as illustrated in the right image of Figure 13 (up to rotation and reflection).

Proof. Suppose the four edges of h are not mapped to distinct edges of $\mathcal{C}$. Then, by Lemma 9, the four edges are not mapped to the same edge, but to two edges forming an L-shape. This effectively amounts to gluing a pair of diagonal grid points of the hole.

Let a, b, c, and d be the four grid points of h, and suppose a and c are mapped to the same corner of $\mathcal{C}$ when folding P into $\mathcal{C}$; see also the left image of Figure 13.

Consider the crease pattern around h. We shall only focus on the angles of the creases and not the type of the fold, as there may be (and will be) other creases in P affecting the type of the creases under our consideration. Observe that the three faces incident to each of the grid points b and d are pairwise perpendicular, they form a corner of a cube. Thus, the creases emanating from b and d are all folded by $\pm 90°$. Further observe that the three unit squares around each of the grid points a and c fold into two faces of a cube, thus, leading to one of the creases being folded by $\pm 90°$ and the other folded by $\pm 180°$. Finally, the two creases folded by $\pm 180°$ are parallel to each other. Indeed, consider the right side of Figure 12. For a crease to form an L-shape one of the two dashed blue lines must fold by $\pm 180°$, which corresponds to two parallel creases in the unfolded state. Therefore, the crease pattern in Figure 13 (left) is the only pattern of creases (up to rotation and reflection) around a non-trivially folded square hole. Figure 13 (right) shows the faces of the corresponding crease pattern, and Figure 11 shows the folding process of this crease pattern. □

With the help of Lemma 10, we can show that several types of polyominoes with unit-square holes do not fold into $\mathcal{C}$.

Theorem 11. *If P is a rectangle with exactly one unit-square hole h, then P does not fold into $\mathcal{C}$.*

Proof. First note that h is folded non-trivially, otherwise P corresponds to a rectangle which does not fold into $\mathcal{C}$ (Corollary 2). Therefore, by Lemma 10, the crease pattern around h is as depicted in Figure 13. Note that, on each side of h, there exists a fold by $\pm 90°$.

Consider the rectangle R obtained by cutting P by the top edge of h and deleting the part below. If R has a height of at least 2, then by Corollary 1, either all vertical or all horizontal creases are folded by $\pm 180°$. In the first case, in particular the creases incident to h are folded by $\pm 180°$. However, this is a contradiction to the crease pattern around h in which each side of h has fold by $\pm 90°$. Consequently, all horizontal edges are folded by $\pm 180°$. This corresponds to folding R on top of the row above h. In particular, P is foldable into $\mathcal{C}$ if and only if the polyomino P' obtained from P by cutting-off all rows above h is foldable. Hence, we consider P'.

Likewise, we treat all other sides of P' and obtain the polyomino P'' consisting of a 3×3-rectangle with a central unit-square hole; see also Figure 13 (right). In particular, P is foldable (if and) only if P'' is foldable into $\mathcal{C}$.

Because h is folded non-trivially, the crease pattern of P'' is given by Figure 13. Note that in the folded state P'' covers only 5 faces and, hence, P'' does not fold into $\mathcal{C}$. □

A similar result holds for rectangular polyominoes with two unit-square holes.

Theorem 12. *A rectangle with exactly two unit-square holes in the same row does not fold into $\mathcal{C}$ if the number of columns between the holes is even.*

Proof. Note that if the polyomino can be folded into $\mathcal{C}$, both holes must be folded non-trivially: If one hole behaves as a square in the folded state, i.e., is folded trivially, the polyomino is effectively reduced to a rectangle with one basic hole. However, by Theorem 11, this does not fold into $\mathcal{C}$. Consequently, both holes are folded non-trivially.

Therefore, by Lemma 10, the crease pattern around the two holes is as depicted in Figure 13. Consider the $3 \times 2k$-rectangle R between the two holes (with $k \geq 1$). By the above observation, at least one horizontal edge of R is folded by $\pm 90°$. Consequently, Corollary 1 implies that all vertical edges are folded by $\pm 180°$. In particular, every unit square of R is mapped to the same face of $\mathcal{C}$ as the leftmost

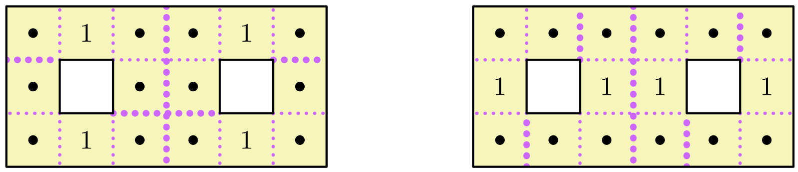

Figure 14: A polyomino that does not fold into a cube.

(or rightmost) unit square in the same row of R. This reduces the polyomino to one with R being a 3×2-rectangle. We will show that the squares of P neighbouring the two holes are not able to cover $\mathcal{C}$, that is, it remains to show that the polyomino P, depicted in Figure 14, does not fold into $\mathcal{C}$.

Consider the left 3×3 block of P. If the two parallel creases folded by $\pm 90°$ are vertical, then the right 3×3 block will also have the two parallel creases folded by $\pm 90°$ run vertical; see Figure 14 (left). Then, the four unit squares above and below the two holes match to the same face on $\mathcal{C}$. Denote it as '1'. Observe that the rest of the unit squares share a grid point with '1' and thus cannot cover the face on $\mathcal{C}$ opposite to '1'.

In the second case, when the two parallel creases folded by $\pm 90°$ of the left block are horizontal, then they extend into the right 3×3 block by Corollary 1. Refer to Figure 14 (right). Then, the four unit squares to the left and to the right of the two holes match to the same face on $\mathcal{C}$, which we denote by '1'. As before, every unit square of P shares a grid point with '1' and thus the face opposite to '1' on $\mathcal{C}$ cannot be covered. □

Remark. Note that the arguments of Lemma 10 and Theorems 11 and 12 extend to an L-slit of size 2, and a U-slit of size 3. The resulting crease patterns are illustrated in Figure 15.

These insights help to obtain the following result:

Theorem 13. *Let P be polyomino with two holes, which are both either a unit square, or an L-slit of size* 2*, or a U-slit of size* 3*, such that (1) P contains all the other cells of the bounding box of the two holes and (2) the number of rows and the number of columns between the holes is at least* 1*. In every folding of P into $\mathcal{C}$, the two holes are not both folded non-trivially.*

Proof. If P contains a unit-square hole that is not folded non-trivially, then, by Lemma 10, the crease pattern in the neighborhood of the hole is as depicted in Figure 13. Likewise, if P contains an L-slit of size 2 or a U-slit of size 3

Figure 15: Crease pattern around an L-slit (left) and a U-slit (right). Numbers indicate the face of the six-sided die in the folded state; thinner lines denote creases folded by ±90°; thicker lines denote creases folded by ±180°; mountain folds are drawn solid/red; and valley folds are drawn dashed/blue.

that is folded non-trivially, the crease pattern in the neighborhood of the hole is as depicted in Figure 15. Note that on each side of the crease patterns in the neighborhood of the holes, there exists a crease folded by ±90°.

We turn the paper such that the left hole is above the right hole as in Figure 16 and consider the rectangular region R to the right of the left hole and above the right hole.

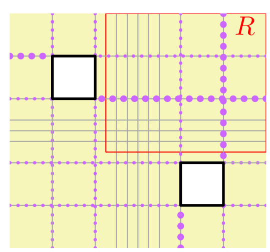

Figure 16: Two unit-square holes with at least one row and column in between, if folded non-trivially imply two perpendicular creases folded by ±90° (drawn as thin dotted lines).

Because each side of the crease patterns in the neighborhood of the holes has a crease folded by ±90° (Lemma 10), R contains a vertical and a horizontal crease folded by ±90°. By Corollary 1, all collinear creases are also folded by ±90°. Hence, there exists a grid point in R for which all incident creases are folded by ±90°, yielding a contradiction to Lemma 7. □

4.2. *Polyominoes with a Single Slit of Size 1*

In the following, we show that a slit hole of size 1 does not help in folding a rectangular polyomino into $\mathcal{C}$. We start with a lemma:

Lemma 14. *In every folding of a polyomino P with a slit hole of size 1, the crease pattern behaves as if the slit hole was nonexistent.*

Proof. To prove the lemma we examine the local neighborhood of the slit and analyze the possible folding patterns we can obtain between adjacent faces. More specifically, we consider the six unit squares A, B, C, D, E and F of P that are incident to the slit hole of size 1 as illustrated in Figure 17. We distinguish two cases: The crease between A and F is folded by $\pm 90°$ or $\pm 180°$.

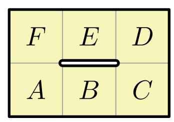

Figure 17: A polyomino with a slit hole of size one.

If the AF crease is folded by $\pm 90°$, we must further distinguish if the EF crease is folded by $\pm 90°$ or by $\pm 180°$. If the EF crease is folded by $\pm 180°$, then the slit edge is mapped to the edge between AF, fixing that B maps to A. Hence, this corresponds to a crease folded by $\pm 90°$ of the slit edge.

By symmetry, we may assume that both the AB crease and the EF crease are folded by $\pm 180°$. This implies that B and E cover the same face in such a way that the top edge of B is mapped to the left edge of E. However, then the bottom left corner of D is also mapped to the top left corner of E. A contradiction. Consequently, this is impossible.

If the AF crease is folded by $\pm 180°$, then A and F cover the same face and, in particular, their left edges are mapped to the same edge such that the top edge of F and the bottom edge of A coincide. This implies that the left edge of E and the left edge of B also coincide such that the top edge of E and the bottom edge of B coincide. This corresponds to crease folded by $\pm 180°$ of the slit edge.

This shows that the slit edge is a crease folded by $\pm 90°$ or by $\pm 180°$. Hence, the crease pattern behaves as if the slit hole was nonexistent. □

Theorem 15. *If P is a rectangle with exactly one slit of size 1, then P does not fold into* $\mathcal{C}$*.*

Proof. By Lemma 14, the crease pattern behaves as if the slit was nonexistent, i.e., as if P was a rectangle. By Corollary 1, all horizontal or vertical creases are folded by $\pm 180°$, reducing P to a rectangle of height or width 1, which does not fold into $\mathcal{C}$. □

Furthermore, we conjecture that the slit of size 1 never is the deciding factor for foldability.

Conjecture 1. *Let polyomino P' be obtained from a polyomino P by adding a slit s of size 1. If P' folds into $\mathcal{C}$, then P folds into $\mathcal{C}$ as well.*

We note that this is not true for arbitrarily large ***polycubes*** (connected three-dimensional polyhedron that are formed by a union of face-adjacent unit cubes on the cube lattice):

Lemma 16. *There exists a polyomino P with a slit s of size 1 and a polycube Q, such that P can be folded into Q, but the polyomino P' without s cannot be folded into Q. That is, for larger polycubes, a slit of size 1 can be the deciding factor for foldability.*

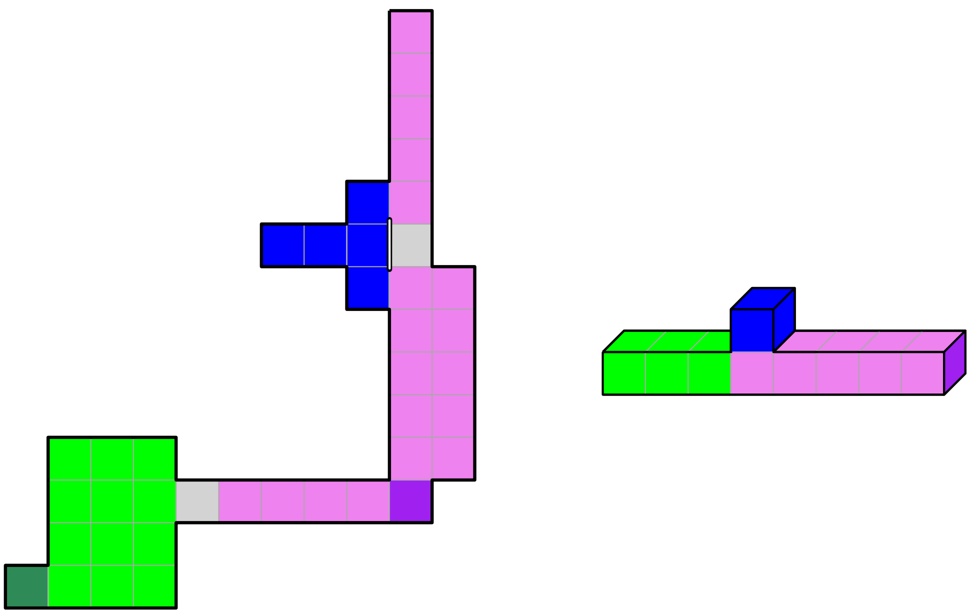

Figure 18: Left: Polyomino P (with slit s), right: polycube Q. Colors of P coincide with the parts of the same color in Q, the light gray unit squares are not mapped to outer faces of Q.

Proof. We consider the polyomino P and the polycube Q from Figure 18. P has 40 unit squares, and Q has 38 faces. Therefore, 38 out of the 40 unit squares will be the faces of Q when folded, hence, at most two unit squares of P may be folded on top of other unit squares to obtain Q.

P contains two rectangular $k \times n$-subpolyominoes with $k, n \geq 2$ that do not contain s: a rectangular 3×4-subpolyomino in the lower left (green) and a rectangular 5×2-subpolyomino (pink). Similar to the proof for Corollary 1, we know that there do not exist two collinear creases in these rectangular subpolyominoes, one of which is folded by $\pm 90°$ and the other by $\pm 180°$. Hence, if we fold a crease in those rectangular subpolyominoes by some angle, all other collinear creases (in the same row or column) are also folded by the same angle. Observe that the surface of Q does not contain any 2×2 flat squares. Hence, for every grid point contained in a rectangle at least the vertical or horizontal creases are folded by some angle.

Assume that we fold the vertical crease of length 5 in the pink rectangular 5×2-subpolyomino by $\pm 180°$. Then 5 unit squares would be folded on top of other unit squares in Q, again a contradiction. If, on the other hand, we fold a horizontal crease of length 2 in the rectangular 5×2-subpolyomino by $\pm 180°$, then all other unit squares need to appear as a face of Q. Similarly, if all of these creases would be $\pm 90°$, again 2 unit squares would be folded on top of other unit squares. However, there are unit squares attached at the bottom and the top of the 5×2-subpolyomino, which in that case cannot cover separate faces (5 unit squares from the pink rectangular 5×2-subpolyomino plus these two adjacent unit squares can cover at most 4 faces of Q), which would yield further overlap, a contradiction. Hence, the crease of length 5 must be $\pm 90°$, thus, this will constitute part of the row of eight unit cubes in Q.

Analogously, assume that we fold any of the horizontal or vertical creases of the green rectangular 3×4-subpolyomino by $\pm 180°$. Hence, 3 or 4 of the unit squares would be folded on top of other unit squares in Q, a contradiction. Consequently, all existing folds must be $\pm 90°$.

Assume that we fold all vertical creases in the green rectangular 3×4 subpolyomino $\pm 90°$. This would yield 3 faces of a tube-like $4 \times 1 \times 1$-polycube for which the 1×1 top and bottom faces and one of the 4×1 faces are missing. However, together with (part of) the pink $5 \times 1 \times 1$ polycube, this would yield a row of nine unit cubes, which cannot be combined for Q. Hence, all horizontal creases must be $\pm 90°$.

Consequently, the green rectangular 3×4-subpolyomino can be folded in a tube-like $3 \times 1 \times 1$-polycube for which the 1×1 top and bottom faces are missing. If we did not use the dark-green leftmost bottom unit square to cover one of these faces, this closing face would need to be a unit square of the remaining $27 (= 40 - (3 \times 4 + 1))$ unit squares, however, then three unit squares must be folded on top of unit squares of the folded 3×4-subpolyomino, a contradiction to the number of faces of Q again.

Hence the $3 \times 1 \times 1$ polycube with one 1×1-face missing (obtained from the green rectangular 3×4-subpolyomino and the adjacent dark-green unit square), must cover the left $3 \times 1 \times 1$-subpolycube of Q.

Then, the only part of P that can be folded into the blue attached unit cube is the blue T-shape.

The vertical unit-square row on top and below that T has length 5, hence, it must cover a part of the right $5 \times 1 \times 1$ of Q (again, otherwise too many unit squares would be folded on top of each other).

We obtain this only when using the slit of size 1 (we push the green 3×4-subpolyomino and the adjacent dark-green unit square through the slit and unfold then again). □

4.3. An Algorithm to Check a Necessary Local Condition for Foldability

Consider the following local condition: let s be a unit square in a polyomino P such that the mapping between grid points of s and corners of a face of $\mathcal{C}$ has been fixed. Then, for every unit square s' adjacent to s, there are two possibilities on how to map its four grid points onto $\mathcal{C}$: the two grid points shared by s and s' must be mapped consistently and for the other two grid points there are two options depending on whether s' is folded by $\pm 90°$ to an adjacent face of $\mathcal{C}$, or whether it is folded by $\pm 180°$ to the same face of $\mathcal{C}$.

The algorithm below checks whether there exists a mapping between all grid points of unit squares of P to corners of $\mathcal{C}$ such that the above condition holds for every pair of adjacent polyomino squares of P.

1. Run a breadth-first search on the polyomino unit squares, starting with the leftmost unit square in the top row of P and continue via adjacent unit squares. This produces a numbering of polyomino unit squares in which each but the first unit square is adjacent to at least one unit square with smaller number. See Figure 19 for an example.
2. Map grid points of the first unit square to the bottom face of $\mathcal{C}$. Extend the mapping one unit square at a time according to the numbering, respecting the local condition (that is, in up to two ways). Track all such partial mappings.

The algorithm is exponential, because unless inconsistencies are produced, the number of possible partial mappings doubles with every polyomino unit square. Nevertheless, it can be used to show non-foldability for small polyominoes: if no consistent mapping exists for a polyomino, then the polyomino cannot be folded onto $\mathcal{C}$. On the other hand, any consistent grid-point mapping covering all faces

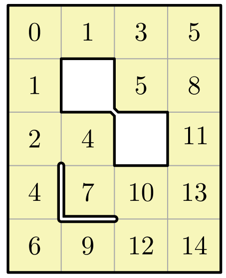

Figure 19: Example of Step 1 of the algorithm. It shows the numbering of polyomino squares produced by the breadth-first search.

of $\mathcal{C}$ obtained by the algorithm that we tried could in practice be turned into a folding. However, we have not been able to prove that this is always the case.

The algorithm above was used to prove that polyominoes in Figure 20 do not fold, as well as it aided us to find the foldings of polyominoes in Figure 8. An implementation of the algorithm is available at the following site `http://github.com/zuzana-masarova/cube-folding`.

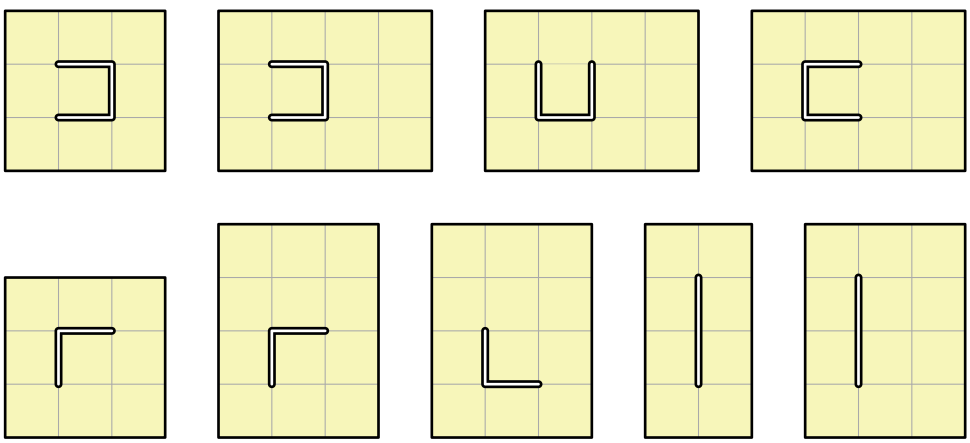

Figure 20: These polyominoes with single L-, U- and straight size-2 slits do not fold into a cube.

5. Conclusion and Open Problems

We showed that, if a polyomino *P* does contain a non-basic hole, then *P* folds into $\mathcal{C}$. Moreover, we showed that a unit-square hole, size-2 slits (straight or L), and a size-3 U-slit sometimes allow for foldability.

Based on the presented results, we created a font of 26 polyominoes with slits that look like each letter of the alphabet, and each fold into $\mathcal{C}$. See Figure 21, and `http://erikdemaine.org/fonts/cubefolding/` for a web app.

We conclude with a list of interesting open problems:

- Does a consistent grid-point mapping output by the algorithm in Section 4.3 imply that the polyomino is foldable? If so, is the folding uniquely determined?
- Is any rectangular polyomino with one L-slit, U-slit, or straight slit of size 2 foldable? Currently, we only know that the small polyominoes in Figure 20 do not fold.
- We considered the existence of only a folded state in the shape of $\mathcal{C}$, but what if we require a continuous folding motion from the unfolded polyomino into $\mathcal{C}$? These two models are known to be equivalent for polygons without holes [13, 3], but equivalence remains an open problem for polygons with holes as in our case.

Acknowledgments

This research was performed in part at the 33rd Bellairs Winter Workshop on Computational Geometry. We thank all other participants for a fruitful atmosphere. H. Akitaya was supported by NSF CCF-1422311 & 1423615. Z. Masárová was partially funded by Wittgenstein Prize, Austrian Science Fund (FWF), grant no. Z 342-N31.

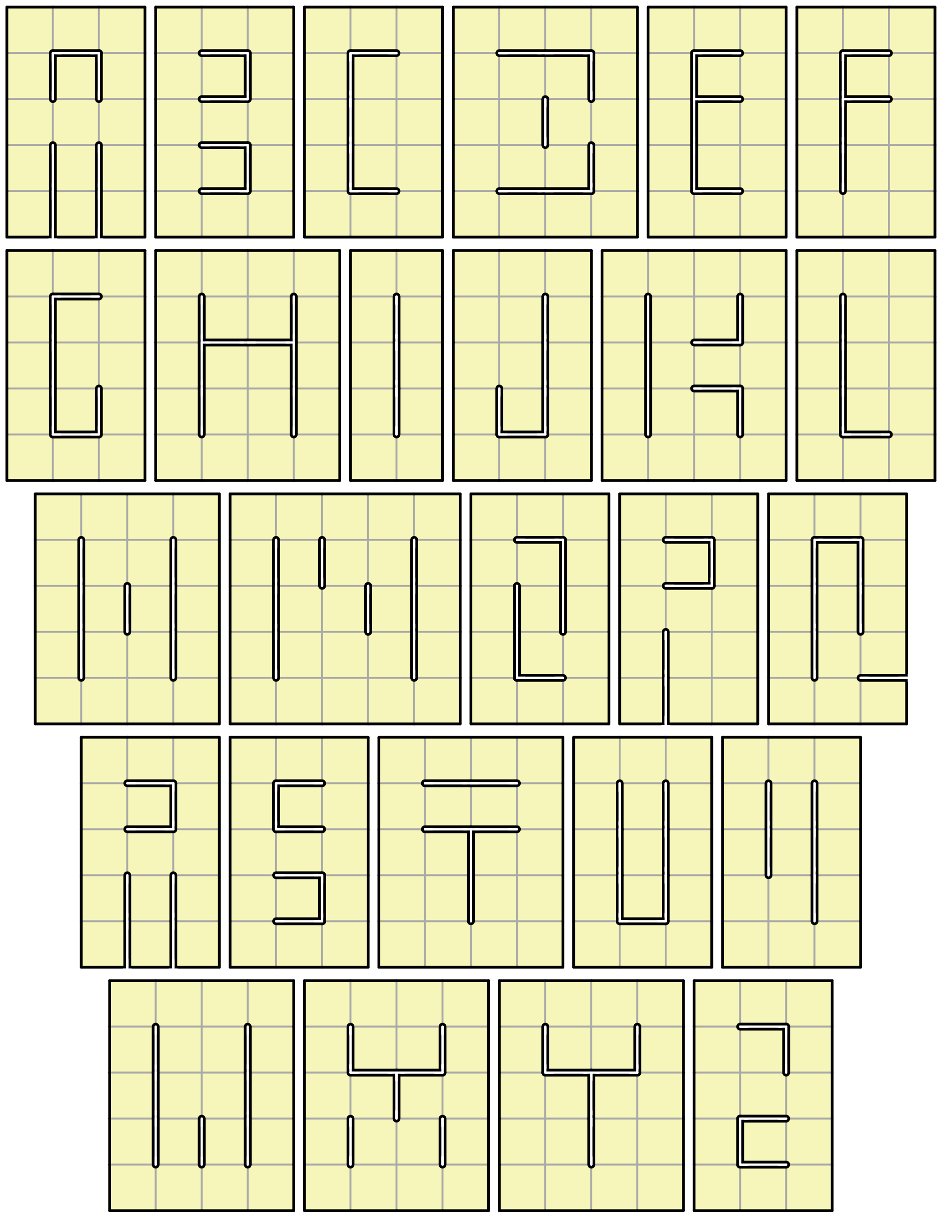

Figure 21: Cube-folding font: the slits representing each letter enable each rectangular puzzle to fold into a cube.

References

[1] O. Aichholzer, H. Akitaya, K. Cheung, E. Demaine, M. Demaine, S. Fekete, L. Kleist, I. Kostitsyna, M. Löffler, Z. Masarova, K. Mundiolva, C. Schmidt, Folding polyominoes with holes into a cube, in: Proc. 31st Can. Conf. Comp. Geom. (CCCG), 2019, pp. 164–170.

[2] N. Beluhov, Cube folding, `https://nbpuzzles.wordpress.com/2014/06/08/cube-folding/` (2014).

[3] E. D. Demaine, J. O'Rourke, Geometric folding algorithms: linkages, origami, polyhedra, Cambridge university press, 2007.

[4] O. Aichholzer, M. Biro, E. D. Demaine, M. L. Demaine, D. Eppstein, S. P. Fekete, A. Hesterberg, I. Kostitsyna, C. Schmidt, Folding polyominoes into (poly)cubes, Int. J. Comp. Geom. & Applic. 28 (03) (2018) 197–226.

[5] K. Y. Czajkowski, E. D. Demaine, M. L. Demaine, K. Eppling, R. Kraft, K. Mundilova, L. Smith, Folding small polyominoes into a unit cube, in: Proc. 32nd Can. Conf. Comp. Geom. (CCCG), 2020.

[6] N. M. Benbernou, E. D. Demaine, M. L. Demaine, A. Lubiw, Universal hinge patterns for folding strips efficiently into any grid polyhedron, Computational Geometry: Theory and Applications (August 2020). `doi:doi:10.1016/j.comgeo.2020.101633`.

[7] G. Aloupis, P. K. Bose, S. Collette, E. D. Demaine, M. L. Demaine, K. Douïeb, V. Dujmović, J. Iacono, S. Langerman, P. Morin, Common unfoldings of polyominoes and polycubes, in: Proc. 9th Int. Conf. Comp. Geom., Graphs & Appl. (CGGA), Vol. 7033 of LNCS, 2011, pp. 44–54. `doi:10.1007/978-3-642-24983-9_5`.

[8] Z. Abel, E. Demaine, M. Demaine, H. Matsui, G. Rote, R. Uehara, Common developments of several different orthogonal boxes, in: Proc. 23rd Can. Conf. Comp. Geom. (CCCG), 2011, pp. 77–82, paper 49.
URL `http://www.cccg.ca/proceedings/2011/papers/paper49.pdf`

[9] J. Mitani, R. Uehara, Polygons folding to plural incongruent orthogonal boxes, in: Proc. 20th Can. Conf. Comp. Geom. (CCCG), 2008, pp. 31–34.

[10] T. Shirakawa, R. Uehara, Common developments of three incongruent orthogonal boxes, Int. J. Comp. Geom. & Applic. 23 (1) (2013) 65–71. doi:http://dx.doi.org/10.1142/S0218195913500040.

[11] R. Uehara, A survey and recent results about common developments of two or more boxes, in: Origami^6: Proc. 6th Int. Meeting Origami in Sci., Math. and Educ. (OSME 2014), Vol. 1, American Mathematical Society, 2014, pp. 77–84.

[12] D. Xu, T. Horiyama, T. Shirakawa, R. Uehara, Common developments of three incongruent boxes of area 30, in: R. Jain, S. Jain, F. Stephan (Eds.), Proc. 12th Ann. Conf. Theory and Applic. of Models of Comput. (TAMC), 2015, pp. 236–247. doi:10.1007/978-3-319-17142-5_21.
URL http://dx.doi.org/10.1007/978-3-319-17142-5_21

[13] E. D. Demaine, J. S. B. Mitchell, Reaching folded states of a rectangular piece of paper, in: Proceedings of the 13th Canadian Conference on Computational Geometry, Waterloo, Canada, 2001, pp. 73–75.