

# Robust Locally Testable Codes and Products of Codes

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## Abstract

We continue the investigation of locally testable codes, i.e., error-correcting codes for whom membership of a given word in the code can be tested probabilistically by examining it in very few locations. We give two general results on local testability: First, motivated by the recently proposed notion of *robust* probabilistically checkable proofs, we introduce the notion of *robust* local testability of codes. We relate this notion to a product of codes introduced by Tanner, and show a very simple composition lemma for this notion. Next, we show that codes built by tensor products can be tested robustly and somewhat locally, by applying a variant of a test and proof technique introduced by Raz and Safra in the context of testing low-degree multivariate polynomials (which are a special case of tensor codes).

Combining these two results gives us a generic construction of codes of inverse polynomial rate, that are testable with poly-logarithmically many queries. We note these locally testable tensor codes can be obtained from *any* linear error correcting code with good distance. Previous results on local testability, albeit much stronger quantitatively, rely heavily on algebraic properties of the underlying codes.

## 1 Introduction

Locally testable codes (LTCs) are error-correcting codes that admit highly efficient probabilistic tests of membership. Specifically, an LTC has a tester that makes a small number of oracle accesses into an oracle representing a given word  $w$ , accepts if  $w$  is a codeword, and rejects with constant probability if  $w$  is far from every codeword. LTCs are combinatorial counterparts of probabilistically checkable proofs (PCPs), and were defined in [18, 25, 2], and their study was revived in [20].

Constructions of locally testable codes typically come in two stages. The first stage is algebraic and gives local tests for algebraic codes, usually based on multivariate polynomials. This is based on a rich collection of results on “linearity testing” or “low-degree testing” [1, 3, 4, 5, 6, 7, 8, 9, 13,

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14, 16, 17, 18, 20, 23, 25]. This first stage either yielded codes of poor rate (mapping  $k$  information symbols to codewords of length  $\exp(k)$ ) as in [14], or yielded codes over large alphabets as in [25]. To reduce the alphabet size, a second stage of “composition” is then applied. In particular, this is done in [20, 13, 11] to get a code mapping  $k$  information bits to codewords of length  $k^{1+o(1)}$ , over the *binary alphabet*. This composition follows the lines of PCP composition introduced in [4], but turns out to be fairly complicated, and in most cases, even more intricate than PCP composition. The one exception is in [20, Section 3], where the composition is simple, but based on very specific properties of the codes used. Thus while the resulting constructions are surprisingly strong, the proof techniques are somewhat complex.

In this paper, we search for simple and general results related to local testing. A generic (non-algebraic) analysis of low-degree tests appears in [19], and a similar approach to PCPs appears in [15]. Specifically, we search for generic (non-algebraic) ways of getting codes, possibly over large alphabets, that can be tested by relatively local tests, as a substitute for algebraic ways. And we look for simpler composition lemmas. We make some progress in both directions. We show that the “tensor product” operation, a classical operation that takes two codes and produces a new one, when applied to linear codes gives codes that are somewhat locally testable (See Theorem 2.6). To simplify the second stage, we strengthen the notion of local testability to a “robust” one. This step is motivated by an analogous step taken for PCPs in [11], but is naturally formulated in our case using the “Tanner Product” for codes [27]. Roughly speaking, a “big” Tanner Product code of block-length  $n$  is defined by a “small” code of block-length  $n' = o(n)$  and a collection of subsets  $S_1, \dots, S_m \subset [n]$ , each of size  $n'$ . A word is in the big code if and only if its projection to every subset  $S_i$  is a word of the small code. Tanner Product codes have a natural local test associated with them: to test if a word  $w$  is a codeword of the big code, pick a random subset  $S_j$  and verify that  $w$  restricted to  $S_j$  is a codeword of the small code. The normal soundness condition would expect that if  $w$  is far from every codeword, then for a constant fraction of such restrictions,  $w$  restricted to  $S_j$  is not a codeword of the small code. Now the notion of robust soundness strengthens this condition further by expecting that if  $w$  is far from every codeword, then many (or most) projections actually lead to words that are *far* from codewords of the small code. In other words, a code is robust if global distance (from the large code) translates into (average) local distance (from the small code). A simple, yet crucial observation is that robust codes compose naturally. Namely, if the small code is itself locally testable by a robust test (with respect to a tiny code, of block-length  $o(n')$ ), then distance from the large code (of block-length  $n$ ) translates to distance from the tiny code, thus reducing query complexity while maintaining soundness. By viewing a tensor product as a robust Tanner product code, we show that a  $(\log N / \log \log N)$ -wise tensor product of *any* linear code of length  $n = \text{poly} \log N$  and relative distance  $1 - \frac{1}{\log N} = 1 - \frac{1}{n^\epsilon}$ , which yields a code of length  $N$  and inverse polynomial rate (i.e. the rate is  $1/N^c$  for  $c$  independent of  $N$ ), that is testable with  $\text{poly}(\log N)$  queries (Theorem 2.7). Once again, while stronger theorems than the above have been known since [6], the generic nature of the result above might shed further light on the notion of local testability.

**Organization.** We give formal definitions and mention our main theorems in Section 2. In Section 3 we analyze the basic tester for tensor product codes. Finally in Section 4 we describe our composition and analyze some tests based on our composition lemma.

## 2 Definitions and Main Results

Throughout this paper  $\Sigma$  will denote a finite alphabet, and in fact a finite field. For positive integer  $n$ , let  $[n]$  denote the set  $\{1, \dots, n\}$ . For a sequence  $x \in \Sigma^n$  and  $i \in [n]$ , we will let  $x_i$  denote the  $i$ th element of the sequence. The Hamming distance between strings  $x, y \in \Sigma^n$ , denoted  $\Delta(x, y)$ , is the number of  $i \in [n]$  such that  $x_i \neq y_i$ . The relative distance between  $x, y \in \Sigma^n$ , denoted  $\delta(x, y)$ , is the ratio  $\Delta(x, y)/n$ .

A code  $C$  of length  $n$  over  $\Sigma$  is a subset of  $\Sigma^n$ . Elements of  $C$  are referred to as codewords. When  $\Sigma$  is a field, one may think of  $\Sigma^n$  as a vector space. If  $C$  is a linear subspace of the vector space  $\Sigma^n$ , then  $C$  is called a linear code. The crucial parameters of a code, in addition to its length and the alphabet, are its dimension (or information length) and its distance, given by  $\Delta(C) = \min_{x \neq y \in C} \{\Delta(x, y)\}$ . A linear code of dimension  $k$ , length  $n$ , distance  $\geq d$  over the alphabet  $\Sigma$  is denoted an  $[n, k, d]_\Sigma$  code. The rate of such a code is the ratio of information to codeword length, i.e.  $k/n$ . For a word  $r \in \Sigma^n$  and a code  $C$ , we let  $\delta_C(r) = \min_{x \in C} \{\delta(r, x)\}$ . We say  $r$  is  $\delta'$ -proximate to  $C$  ( $\delta'$ -far from  $C$ , respectively) if  $\delta_C(r) \leq \delta'$  ( $\delta_C(r) \geq \delta'$ , respectively).

Throughout this paper, we will be working with infinite families of codes, where their performance will be measured as a function of their length.

**Definition 2.1 (Tester)** *A tester  $T$  with query complexity  $q(\cdot)$  is a probabilistic oracle machine that when given oracle access to a string  $r \in \Sigma^n$ , makes  $q(n)$  queries to the oracle for  $r$  and returns an accept/reject verdict. We say that  $T$  tests a code  $C$  if whenever  $r \in C$ ,  $T$  accepts with probability one; and when  $r \notin C$ , the tester rejects with probability at least  $\delta_C(r)/2$ . A code  $C$  is said to be locally testable with  $q(n)$  queries if there is a tester for  $C$  with query complexity  $q(n)$ .*

When referring to oracles representing vectors in  $\Sigma^n$ , we emphasize the queries by denoting the response of the  $i$ th query by  $r[i]$ , as opposed to  $r_i$ . Through this paper we consider only non-adaptive testers, i.e., testers that use their internal randomness  $s$  to generate  $q$  queries  $i_1, \dots, i_q \in [n]$  and a predicate  $P : \Sigma^q \rightarrow \{0, 1\}$  and accept iff  $P(r[i_1], \dots, r[i_q]) = 1$ .

Our next definition is based on the notion of Robust PCP verifiers introduced by [11]. We need some terminology first.

Note that a tester  $T$  has two inputs: an oracle for a received vector  $r$ , and a random string  $s$ . On input the string  $s$  the tester generates queries  $i_1, \dots, i_q \in [n]$  and fixes circuit  $C = C_s$  and accepts if  $C(r[i_1], \dots, r[i_q]) = 1$ .

**Definition 2.2 (Robust Tester)** *For oracle  $r$  and random string  $s$ , define the robustness of the tester  $T$  on  $r, s$ , denoted  $\rho^T(r, s)$ , to be the minimum, over strings  $x$  satisfying  $C(x) = 1$ , of relative distance of  $\langle r[i_1], \dots, r[i_q] \rangle$  from  $x$ . The expected robustness of  $T$  on  $r$  is the following quantity,*

$$\rho^T(r) \stackrel{\text{def}}{=} \mathbf{E}_s[\rho^T(r, s)]$$

When  $T$  is clear from context, we skip the superscript.

A tester  $T$  is said to be  $\alpha$ -robust for a code  $C$  if for every  $r \in C$ , the tester accepts with probability one, and for every  $r \in \Sigma^n$ ,  $\rho^T(r) \geq \alpha \cdot \delta_C(r)$ .

Having a robust tester for a code  $C$  implies the existence of a tester for  $C$ , as illustrated by the following proposition.

**Proposition 2.3** *If a code  $C$  has a  $\alpha$ -robust tester  $T$  for  $C$  making  $q$  queries, then it is locally testable with  $O(q/\alpha)$  queries.*

**Proof:** Let  $c = \lceil \alpha^{-1} \rceil$ . The local tester  $T'$  for  $C$  is obtained by invoking  $T$   $c$  times and accepting if all invocations accept. Consider a word  $r$  with  $\delta_C(r) = \delta$ . For at least  $\delta/c \leq \alpha \cdot \delta$  fraction of the choices of random strings  $s$  of  $T$ , it must be that  $\rho^T(r, s) > 0$  and  $T$  rejects. Thus the probability that  $T'$  does not reject in any of the  $c$  repetitions is at most

$$\begin{aligned} (1 - \delta/c)^c &\leq 1 - c(\delta/c) + \binom{c}{2}(\delta/c)^2 \quad (\text{By Inclusion-Exclusion}) \\ &\leq 1 - \delta + c^2/2(\delta/c)^2 \\ &= 1 - \delta + \delta^2/2 \\ &\leq 1 - \delta + \delta/2 \\ &= 1 - \delta/2 \end{aligned}$$

Thus words at distance  $\delta$  from codewords are rejected with probability at least  $\delta/2$ .  $\blacksquare$

The previous proposition shows that large robustness leads to small query complexity. However, there is a limit to the size of the robustness parameter as shown in the next claim.

**Proposition 2.4** *If  $T$  is a  $\alpha$ -robust tester for a linear code  $C \subset \Sigma^n$  with minimal Hamming distance at least two, then  $\alpha \leq 1$ .*

**Proof:** As shown in [12], a tester  $T$  for a linear code can be assumed (w.l.o.g.) non-adaptive, i.e. the set of queries performed by  $T$  does not depend on the received word  $r$  (only on the randomness  $s$ ). Assuming non-adaptivity, let  $T_1, \dots, T_S$  be the set of possible tests performed by  $T$ , let  $p_j$  be the probability  $T_j$  is performed, and let  $q_j$  be the query complexity of  $T_j$ . Let  $S_i$  be the set of tests that query  $i \in [n]$  and let  $\text{wt}(i) = \sum_{j \in S_i} p_j/q_j$  be the weight of  $i \in [n]$ . There must be some  $i$  with weight  $\leq 1/n$  because the sum of weights is one. Look at the word  $r$  that is zero everywhere but on the  $i^{\text{th}}$  coordinate, where it is one. On the one hand  $\delta_C(r) = 1/n$ , because  $C$  is a linear code of minimal distance  $> 1$ . On the other hand, the robustness of  $\rho^T(r) = \text{wt}(i) \leq 1/n$ . Thus, the robustness parameter is at most one.  $\blacksquare$

The main results of this paper focus on robust local testability of certain codes. For the first result, we need to describe the tensor product of codes.

**Tensor Products and Local Tests** Recall that an  $[n, k, d]_\Sigma$  linear code  $C$  may be represented by a  $k \times n$  matrix  $M$  over  $\Sigma$  (so that  $C = \{xM \mid x \in \Sigma^k\}$ ). Such a matrix  $M$  is called a **generator** of  $C$ . Given an  $[n_1, k_1, d_1]_\Sigma$  code  $C_1$  with generator  $M_1$  and an  $[n_2, k_2, d_2]_\Sigma$  code  $C_2$  with generator  $M_2$ , their **tensor product** (cf. [22], [26, Lecture 6, Section 2.4]), denoted  $C_1 \otimes C_2 \subseteq \Sigma^{n_2 \times n_1}$ , is the

code whose codewords may be viewed as  $n_2 \times n_1$  matrices given explicitly by the set  $\{M_2^T X M_1 | X \in \Sigma^{k_2 \times k_1}\}$ . It is well-known that  $C_1 \otimes C_2$  is an  $[n_1 n_2, k_1 k_2, d_1 d_2]_\Sigma$  code.

Tensor product codes are interesting to us in that they are a generic construction of codes with “non-trivially” local redundancy. To elaborate, every linear code of dimension  $k$  does have redundancies of size  $O(k)$ , i.e., there exist subsets of  $t = O(k)$  coordinates where the code does not take all possible  $\Sigma^t$  possible values. But such redundancies are not useful for constructing local tests; and unfortunately generic codes of length  $n$  and dimension  $k$  may not have any redundancies of length  $o(k)$ . However, tensor product codes are different in that the tensor product of an  $[n, k, d]_\Sigma$  code  $C$  with itself leads to a code of dimension  $k^2$  which is much larger than the size of redundancies which are  $O(k)$ -long, as asserted by the following proposition.

**Proposition 2.5** *A matrix  $r \in \Sigma^{n_2 \times n_1}$  is a codeword of  $C_1 \otimes C_2$  if and only if every row is a codeword of  $C_1$  and every column is a codeword of  $C_2$ .*

In addition to being non-trivially local, the constraints enumerated above are also redundant, in that it suffices to insist that all columns are codewords of  $C_2$  and only  $k_2$  (prespecified) rows are codewords of  $C_1$ . Thus the insistence that other rows ought to be codewords of  $C_1$  is redundant, and leads to the hope that the tests may be somewhat robust. Indeed we may hope that the following might be a robust test for  $C_1 \otimes C_2$ .

**Product Tester:** Pick  $b \in \{1, 2\}$  at random and  $i \in [n_b]$  at random. Verify that  $r$  with  $b$ th coordinate restricted to  $i$  is a codeword of  $C_{3-b}$ .

While it is possible to show that the above is a reasonable tester for  $C_1 \otimes C_2$  for any pair of codes  $C_1, C_2$ , the query complexity of the test is  $\max\{n_1, n_2\}$ , which is quite high. If the test were *robust*, there would be ways of reducing this query complexity (as we will see later). Unfortunately, this is not the case. Indeed, Valiant [28] recently showed that for every  $\epsilon > 0$  there exist (arbitrarily large) codes  $C_1, C_2$  such that the product tester for  $C_1 \otimes C_2$  is not  $\epsilon$ -robust.

Instead, we consider higher products of codes, and give a tester based on an idea from the work of Raz and Safra [24]. Specifically, we let  $C^m$  denote the code  $\underbrace{C \otimes \cdots \otimes C}_m$ . We consider the following test for this code:

**$m$ -Product Tester:** Pick  $b \in [m]$  and  $i \in [n]$  independently and uniformly at random. Verify that  $r$  with  $b$ th coordinate restricted to  $i$  is a codeword of  $C^{m-1}$ .

Note that this tester makes  $N^{1-\frac{1}{m}}$  queries to test a code of length  $N = n^m$ . So its query complexity gets worse as  $m$  increases. However, we are only interested in the performance of the test for small  $m$  (specifically  $m = 3, 4$ ). We show that the test is a robust tester for  $C^m$  for every  $m \geq 3$ . Specifically, we show

**Theorem 2.6** *For a positive integer  $m$  and  $[n, k, d]_\Sigma$ -code  $C$ , such that  $(\frac{d-1}{n})^m \geq \frac{7}{8}$ ,  $m$ -Product Tester is  $2^{-16}$ -robust for  $C^m$ .*

This theorem is proven in Section 3. Note that the robustness is a constant, and the theorem only needs the fractional distance of  $C$  to be sufficiently large as a function of  $m$ . In particular a fractional distance of  $1 - \frac{1}{O(m)}$  suffices. Note that such a restriction is needed even to get the fractional distance of  $C^m$  to be constant.

The tester however makes a lot of queries, and this might seem to make this result uninteresting (and indeed one doesn't have to work so hard to get a non-robust tester with such query complexity). However, as we note next, the query complexity of robust testers can be reduced significantly under some circumstances. To describe this we need to revisit a construction of codes introduced by Tanner [27].

**Tanner Products and Robust Testing** The robustness of the  $m$ -Product Tester above seems to be naturally related to the fact that the tester's predicates are testing if the queried points themselves belong to a smaller code. (In the case of the  $m$ -Product Tester, it verifies that the symbols it reads give a codeword of the code  $C^{m-1}$ .) The notion that a bigger code (such as  $C^m$ ) may be specified by requiring that certain projections of a word fall in a smaller code (such as  $C^{m-1}$ ) is not a novel one. Indeed this idea goes back to the work of Tanner [27], who defined this notion in its full generality and considered big codes obtained by a "product" of a bipartite graph with a small code. This notion is commonly referred to in the literature as the Tanner Product, and we define it next.

For integers  $(n, m, t)$  an  $(n, m, t)$ -ordered bipartite graph is given by  $n$  left vertices  $[n]$ , and  $m$  right vertices, where each right vertex has degree  $t$  and the neighborhood of a right vertex  $j \in [m]$  is ordered and given by a sequence  $\ell_j = \langle \ell_{j,1}, \dots, \ell_{j,t} \rangle$  with  $\ell_{j,i} \in [n]$ .

A Tanner Product Code (TPC), is specified by an  $[n, m, t]$  ordered bipartite graph  $G$  and a code  $C_{\text{small}} \subseteq \Sigma^t$ . The product code, denoted  $\text{TPC}(G = \{\ell_1, \dots, \ell_m\}, C_{\text{small}}) \subseteq \Sigma^n$ , is the set

$$\{r \in \Sigma^n \mid r|_{\ell_j} \stackrel{\text{def}}{=} \langle r_{\ell_{j,1}}, \dots, r_{\ell_{j,t}} \rangle \in C_{\text{small}}, \forall j \in [m]\}.$$

Notice that the Tanner Product naturally suggests a test for a code. "Pick a random right vertex  $j \in [m]$  and verify that  $r|_{\ell_j} \in C_{\text{small}}$ ." Associating this test with such a pair  $(G, C_{\text{small}})$ , we say that the pair is  $\alpha$ -robust if the associated test is a  $\alpha$ -robust tester for  $\text{TPC}(G, C_{\text{small}})$ .

The importance of this representation of tests comes from the composability of robust tests coming from Tanner Product Codes. Suppose  $(G, C_{\text{small}})$  is  $\alpha$ -robust and  $C_{\text{small}}$  is itself a Tanner Product Code,  $\text{TPC}(G', C_{\text{small}}')$  where  $G'$  is an  $(d, m', t')$ -ordered bipartite graph and  $(G', C_{\text{small}}')$  is  $\alpha'$ -robust. Then  $\text{TPC}(G, C_{\text{small}})$  has an  $\alpha \cdot \alpha'$ -robust tester that makes only  $t'$  queries. (This fact is completely straightforward and proven in Lemma 4.1.)

This composition is especially useful in the context of tensor product codes. For instance, the tester for  $C^4$  is of the form  $(G, C^3)$ , while  $C^3$  has a robust tester of the form  $(G', C^2)$ . Putting them together gives a tester for  $C^4$ , where the tests verify appropriate projections are codewords of  $C^2$ . The test itself is not surprising, however the ease with which the analysis follows is nice. (See Lemma 4.2.) Now the generality of the tensor product tester comes in handy as we let  $C$  itself be  $C'^2$  to see that we are now testing  $C'^8$  where tests verify some projections are codewords of  $C'^4$ . Again composition allows us to reduce this to a  $C'^2$ -test. Carrying on this way we see that we can test any code of the form  $C^{2^t}$  by verifying certain projections are codewords of  $C^2$ . This leads to a

simple proof of the following theorem about the testability of tensor product codes. (Throughout this paper we do not attempt to optimize constants).

**Theorem 2.7** *Let  $\{C_i\}_i$  be any infinite family of codes with  $C_i$  a  $[n_i, k_i, d_i]_{\Sigma_i}$  code. Further, let  $t_i$  be a sequence of integers such that  $m_i = 2^{t_i}$  satisfies  $d_i/n_i \geq 1 - \frac{1}{7m_i}$ . Then the sequence of codes  $\{C'_i = C_i^{m_i}\}_i$  is a sequence of  $[n_i^{m_i}, k_i^{m_i}, n_i^{m_i}/2]_{\Sigma_i}$ -codes, where  $C'_i$  is locally testable with query complexity  $q_i = n_i^2 \cdot m_i^{O(1)}$ .*

*In particular, if  $n_i \leq \min\{p(k_i), \hat{p}(m_i)\}$  for some polynomials  $p(\cdot), \hat{p}(\cdot)$ , then  $\{C'_i\}_i$  is a sequence of codes with inverse polynomial rate (and relative distance half) that is locally testable with polylogarithmic query complexity.*

This theorem is proven in Section 4. We remark that it is possible to get code families  $C_i$  such as above using Reed-Solomon codes, as well as algebraic-geometric codes.

### 3 Testing Tensor Product Codes

In this section we prove Theorem 2.6. We first reformulate this theorem in the language of Tanner products.

Let  $G_m^n$  denote the graph that corresponds to the tests of  $C^m$  by the  $m$ -Product Tester, where  $C \subseteq \Sigma^n$ . Namely  $G_m^n$  has  $n^m$  left vertices labeled by elements of  $[n]^m$ . It has  $m \cdot n$  right vertices labeled  $(b, i)$  with  $b \in [m]$  and  $i \in [n]$ . Vertex  $(b, i)$  is adjacent to all vertices  $(i_1, \dots, i_m)$  such that  $i_b = i$ . The statement of Theorem 2.6 is equivalent to the statement that  $(G_m^n, C^{m-1})$  is  $2^{-16}$ -robust, provided  $\left(\frac{d-1}{n}\right)^m \geq \frac{7}{8}$ . The completeness of the theorem follows from Proposition 2.5, which implies  $C^m = \text{TPC}(G_m^n, C^{m-1})$ . For the soundness, we first introduce some notation.

Consider the code  $C_1 \otimes \dots \otimes C_m$ , where  $C_i = [n_i, k_i, d_i]_{\Sigma}$  code. Notice that codewords of this code lie in  $\Sigma^{n_1 \times \dots \times n_m}$ . The coordinates of strings in  $\Sigma^{n_1 \times \dots \times n_m}$  are themselves  $m$ -dimensional vectors over the integers (from  $[n_1] \times \dots \times [n_m]$ ). For  $r \in \Sigma^{n_1 \times \dots \times n_m}$  and  $i_1, \dots, i_m$  with  $i_j \in [n_j]$ , let  $r[i_1, \dots, i_m]$  denote the  $\langle i_1, \dots, i_m \rangle$ -th coordinate of  $r$ . For  $b \in [m]$ , and  $i \in [n_b]$ , let  $r_{b,i} \in \Sigma^{n_1 \times \dots \times n_{b-1} \times n_{b+1} \times \dots \times n_m}$  be the vector obtained by projecting  $r$  to coordinates whose  $b$ th coordinate is  $i$ , i.e.,  $r_{b,i}[i_1, \dots, i_{m-1}] = r[i_1, \dots, i_{b-1}, i, i_b, \dots, i_{m-1}]$ .

The following simple property about tensor product codes will be needed in our proof.

**Proposition 3.1** *For  $b \in \{1, \dots, m\}$  let  $C_b$  be an  $[n_b, k_b, d_b]_{\Sigma}$  code, and let  $I_b$  be a set of cardinality at least  $n_b - d_b + 1$ . Let  $C'_b$  be the code obtained by the projection of  $C_b$  to  $I_b$ . Then every codeword  $c'$  of  $\mathcal{C}' = C'_1 \otimes \dots \otimes C'_m$  can be extended to a unique codeword  $c$  of  $\mathcal{C} = C_1 \otimes \dots \otimes C_m$ .*

**Proof:** The projection of  $C_b$  to  $C'_b$  is bijective. It is surjective because it is a projection, and it is injective because  $|I_b| > n_b - d_b$ . So, the projection of  $\mathcal{C}$  to  $\mathcal{C}'$  is a bijection, because both codes are of dimension  $\prod_{b=1}^m d_b$ . Thus, every word in  $\mathcal{C}'$  has a unique preimage in  $\mathcal{C}$ .  $\blacksquare$

Recall that the  $m$ -Product tester picks a random  $b \in [m]$  and  $i \in [n]$  and verifies that  $r_{b,i} \in C^{m-1}$ . Let  $\rho(r, (b, i))$  denote the distance of the view of this tester when accessing oracle  $r$  on random

string  $(b, i)$ . Note that  $\rho(r, (b, i)) = \delta_{C^{m-1}}(r_{b,i})$ . Let  $\rho(r) = \mathbf{E}_{b,i}[\delta_{C^{m-1}}(r_{b,i})]$ . We wish to show for every  $r$  that  $\rho(r) \geq 2^{-16} \cdot \delta_{C^m}(r)$  or equivalently  $\delta_{C^m}(r) \leq 2^{16} \cdot \rho(r)$ .

We start by first getting a crude upper bound on the proximity of  $r$  to  $C^m$  and then we use the crude bound to get a tighter relationship. To get the crude bound, we first partition the random strings into two classes: those strings  $(b, i)$  for which  $\rho(r, (b, i))$  is large, and those for which it is small. To obtain our partition we need the following definition.

**Definition 3.2 (Soundness Error)** For  $r \in \Sigma^{n^m}$  and threshold  $\tau \in [0, 1]$ , define the  $\tau$ -soundness-error of  $r$  to be the probability that  $\delta_{C^{m-1}}(r_{b,i}) > \tau$ , when  $b \in [m]$  and  $i \in [n]$  are chosen uniformly and independently.

We start by showing  $r$  is somewhat close to some codeword of  $C^m$ .

**Lemma 3.3** If the  $\tau$ -soundness-error of  $r$  is  $\epsilon$  for  $\tau + 2\epsilon \leq \frac{1}{12} \cdot \left(\frac{d-1}{n}\right)^m$ , then  $\delta_{C^m}(r) \leq 16 \cdot \left(\frac{n}{d}\right)^{m-1} \cdot (\tau + \epsilon)$ .

Note that for  $\rho = \rho(r)$ , the  $\sqrt{\rho}$ -soundness error of  $r$  is at most  $\sqrt{\rho}$ . Thus, setting  $\tau = \epsilon = \sqrt{\rho}$  in Lemma 3.3 shows  $r$  is  $O(\sqrt{\rho})$ -close to some codeword of  $C^m$ .

**Proof:** For every  $i \in [n]$  and  $b \in [m]$ , fix  $c_{b,i}$  to be a closest codeword from  $C^{m-1}$  to  $r_{b,i}$ . We follow the proof outline of Raz & Safra [24] which when adapted to our context goes as follows: (1) Given a vector  $r$  and a set of codewords  $\{c_{b,i} \in C^{m-1} : b \in [m], i \in [n]\}$ , we define an ‘‘inconsistency’’ graph  $G$ . (Note that this graph is *not* the same as the graph  $G_m^m$  that defines the test being analyzed. In particular  $G$  is related to the word  $r$  being tested.) (2) We show that the existence of a large independent set in this graph  $G$  implies the proximity of  $r$  to a codeword of  $C^m$  (i.e.,  $\delta_{C^m}(r)$  is small). (3) We show that this inconsistency graph is sparse if the  $\tau$ -soundness-error is small. (4) We show that the distance of  $C$  forces the graph to be special in that every edge is incident to at least one vertex whose degree is large.

**Definition of  $G$ .** The vertices of  $G$  are indexed by pairs  $(b, i)$  with  $b \in [m]$  and  $i \in [n]$ . Vertex  $(b_1, i_1)$  is adjacent to  $(b_2, i_2)$  if *at least* one of the following conditions hold:

1.  $\delta_{C^{m-1}}(r_{b_1, i_1}) > \tau$ .
2.  $\delta_{C^{m-1}}(r_{b_2, i_2}) > \tau$ .
3.  $b_1 \neq b_2$  and  $c_{b_1, i_1}$  and  $c_{b_2, i_2}$  are inconsistent, i.e., there exists some element  $j = \langle j_1, \dots, j_m \rangle \in [n]^m$ , with  $j_{b_1} = i_1$  and  $j_{b_2} = i_2$  such that  $c_{b_1, i_1}[j^{(1)}] \neq c_{b_2, i_2}[j^{(2)}]$ , where  $j^{(e)} \in [n]^{m-1}$  is the vector  $j$  with its  $b_e$ th coordinate deleted.



**Independent sets of  $G$  and proximity of  $r$ .** It is clear that  $G$  has  $mn$  vertices. We claim next that if  $G$  has an independent set  $I$  of size at least  $m(n-d) + d + 1$  then  $r$  has distance at most  $1 - (|I|/(mn))(1 - \tau)$  to  $C^m$ .

Consider an independent set  $I = I_1 \cup \dots \cup I_m$  in  $G$  with  $I_b$  of size  $n_b$  being the set of vertices of the form  $(b, i), i \in [n]$ . W.l.o.g. assume  $n_1 \geq \dots \geq n_m$ . Then, we have  $n_1, n_2 > n - d$  (or else even if  $n_1 = n$  and  $n_2 = n - d$  we'd only have  $\sum_b n_b \leq n + (m-1)(n-d)$ ). We consider the partial vector  $r' \in \Sigma^{I_1 \times n \times \dots \times n}$  defined as  $r'[i, j_2, \dots, j_m] = c_{1,i}[j_2, \dots, j_m]$  for  $i \in I_1$ , and  $j_2, \dots, j_m \in [n]$ . We show that  $r'$  can be extended into a codeword of  $C^m$  and that the extended word is close to  $r$  and this will give the claim.

First, we show that any extension of  $r'$  is close to  $r$ : This is straightforward since on each coordinate  $i \in I_1$ , we have  $r$  agrees with  $r'$  on  $1 - \tau$  fraction of the points. Furthermore  $I_1/n$  is at least  $|I|/(mn)$  (since  $n_1$  is the largest). So we have that  $r'$  is at most  $1 - (|I|/(mn))(1 - \tau)$  far from  $r$ .

Now we prove that  $r'$  can be extended into a codeword of  $C^m$ . Let  $C_b = C|_{I_b}$  be the projection (puncturing) of  $C$  to the coordinates in  $I_b$ . Let  $r''$  be the projection of  $r'$  to the coordinates in  $I_1 \times I_2 \times [n] \times \dots \times [n]$ . We will argue below that  $r''$  is a codeword of  $C_1 \otimes C_2 \otimes C^{m-2}$ , by considering its projection to axis-parallel lines and claiming all such projections yield codewords of the appropriate code. Note first that the restriction of  $r'$  to any line parallel to the  $b$ -th axis is a codeword of  $C$ , for every  $b \in \{2, \dots, m\}$ , since  $r'_{1,i}$  is a codeword of  $C^{m-1}$  for every  $i \in I_1$ . Thus this continues to hold for  $r''$  (except that now the projection to a line parallel to the 2nd coordinate axis is a codeword of  $C_2$ ). Finally, consider a line parallel to the first axis, given by restricting the other coordinates to  $\langle i_2, \dots, i_m \rangle$ , with  $i_2 \in I_2$ . We claim that for every  $i_1 \in I_1$ ,  $r''[i_1, \dots, i_m] = c_{2,i_2}[i_1, \dots, i_m]$ . This follows from the fact that the vertices  $(1, i_1)$  and  $(2, i_2)$  are not adjacent to each other and thus implying that  $c_{1,i_1}$  and  $c_{2,i_2}$  are consistent with each other. We conclude that the restriction of  $r''$  to every axis parallel line is a codeword of the appropriate code, and thus (by Proposition 2.5),  $r''$  is a codeword of  $C_1 \otimes C_2 \otimes C^{m-2}$ . Now applying Proposition 3.1 to the code  $C_1 \otimes C^{m-1}$  and its projection  $C_1 \otimes C_2 \otimes C^{m-2}$  we get that there exists a unique extension of  $r''$  into a codeword  $c'$  of the former. We claim this extension is exactly  $r'$  since for every  $i \in I_1$ ,  $c'_{1,i}[j, k] = r'[i, j, k]$ . Finally applying Proposition 3.1 one more time, this time to the code  $C^m$  and its projection  $C_1 \otimes C^{m-1}$ , we find that  $r' = c'$  can be extended into a codeword of the former. This concludes the proof of this claim.

**Density of  $G$ .** We now see that the small  $\tau$ -soundness-error of the test translates into a small density  $\gamma$  of edges in  $G$ . Below, we refer to pairs  $(b, i)$  with  $b \in [m]$  and  $i \in [n]$  as “planes” (since they refer to  $(m-1)$ -dimensional planes in  $[n]^m$ ) and refer to elements of  $[n]^m$  as “points”. We say a point  $p = \langle p_1, \dots, p_m \rangle$  lies on a plane  $(b, i)$  if  $p_b = i$ . Now consider the following test: Pick a random plane  $(b_1, i_1)$  and a random point  $p$  on it and verify  $c_{b_1, i_1}$  is consistent with  $r[p]$ . Let  $\kappa$  denote the rejection probability of this test. We bound  $\kappa$  from both sides.

For the sake of analysis, we consider the following test: Pick two random planes  $(b_1, i_1)$  and  $(b_2, i_2)$  subject to the constraint  $b_1 \neq b_2$  and pick a random point  $p$  in the intersection of the two planes and verify that  $c_{b_1, i_1}$  is consistent with  $r[p]$ . It is easy to see this test is identical to the previous one, and we analyze this latter test.

On the one hand we have that the rejection probability is at least the probability that we pick two planes that are  $\tau$ -proximate to codewords of  $C^{m-1}$  and incident to each other in  $G$  (which is at

least  $\frac{m\gamma}{m-1} - 2\epsilon$ ) and the probability that we pick a point on the intersection at which the two plane codewords disagree (at least  $(d/n)^{m-2}$ ), times the probability that the codeword that disagrees with the point function is the first one (which is at least  $1/2$ ). Thus we get  $\kappa \geq \frac{d^{m-2}}{2(n)^{m-2}} \left( \frac{m\gamma}{m-1} - 2\epsilon \right)$ .

On the other hand we have that in order to reject it must be the case that either  $\delta_{C^{m-1}}(r_{b_1, i_1}) > \tau$  (which happens with probability at most  $\epsilon$ ) or  $\delta_{C^{m-1}}(r_{b_1, i_1}) \leq \tau$  and  $p$  is such that  $r_{b_1, i_1}$  and  $c_{b_1, i_1}$  disagree at  $p$  (which happens with probability at most  $\tau$ ). Thus we have  $\kappa \leq \tau + \epsilon$ . Putting the two together we have  $\gamma \leq \frac{m-1}{m} \left( 2\epsilon + \frac{2n^{m-2}}{d^{m-2}} (\tau + \epsilon) \right)$ .

**Structure of  $G$ .** Next we note that every edge of  $G$  is incident to at least one high-degree vertex. Consider a pair of planes that are adjacent to each other in  $G$ . If either of the vertices is not  $\tau$ -robust, then it is adjacent to every vertex of  $G$ . So assume both are  $\tau$ -robust.

W.l.o.g., let these be the vertices  $(1, i)$  and  $(2, j)$ . Thus the codewords  $c_{1, i}$  and  $c_{2, j}$  disagree on the  $(m-2)$ -dimensional surface with the first two coordinates restricted to  $i$  and  $j$  respectively. Now let  $S = \{ \langle i, j, k_3, \dots, k_m \rangle \mid c_{1, i}[j, k_3, \dots, k_m] \neq c_{2, j}[i, k_3, \dots, k_m] \}$  be the set of disagreeing tuples on this surface. By the distance of  $C^{m-2}$  we know  $|S| \geq d^{m-2}$ . Furthermore, by construction of  $C^{m-2}$ , for every  $b \in \{3, \dots, m\}$  there exist at least  $d$  points  $k^{(1)}, \dots, k^{(d)} \in S$  that differ on their  $b^{\text{th}}$  coordinate (i.e. the  $b^{\text{th}}$  coordinate of  $k^{(\ell)}$  differs from the  $b^{\text{th}}$  coordinate of  $k^{(t)}$  for  $1 \leq \ell < t \leq d$ ). For each such  $k^{(\ell)}$ , with  $b^{\text{th}}$  coordinate equal  $k_b^{(\ell)}$ , the vertex  $(b, k_b^{(\ell)})$  must be adjacent to at least one of  $(1, i)$  or  $(2, j)$ . This is because  $k^{(\ell)}$  is a point on the plane  $(b, k_b^{(\ell)})$  and  $(b_1, i), (b_2, i)$  disagree on  $k^{(\ell)}$ . We conclude the sum of the degrees of  $(1, i)$  and  $(2, j)$  is at least  $(m-2)d$ , and so at least one of them has degree at least  $(m-2)d/2$ .

**Putting it together.** From the last paragraph above, we have that the set of vertices of degree less than  $(m-2)d/2$  form an independent set in the graph  $G$ . The fraction of vertices of degree at least  $(m-2)d/2$  is at most  $2(\gamma mn)/((m-2)d)$ . Thus we get that if  $mn \cdot (1 - 2(\gamma mn)/((m-2)d)) \geq m(n-d) + d + 1$ , then  $r$  is  $\delta$ -proximate to  $C^m$  for  $\delta \leq \tau + (1-\tau) \cdot 2(\gamma mn)/((m-2)d)$ . The lemma now follows by simplifying the expressions above, using the upper bound on  $\gamma$  derived earlier. Details below.

We first focus on the condition  $|I| \geq m(n-d) + d + 1$ . It suffices to prove that

$$\begin{aligned}
mn \cdot (1 - 2(\gamma mn)/((m-2)d)) &\geq m(n-d) + d + 1 \\
\Leftrightarrow (m-1)d - 1 &\geq 2 \frac{\gamma m^2 n^2}{(m-2)d} \\
\Leftrightarrow (m-1)(d-1) &\geq 2 \frac{\gamma m^2 n^2}{(m-2)d} \\
\Leftrightarrow (m-1)(d-1) &\geq 2 \frac{m^2 n^2}{(m-2)d} \cdot \frac{m-1}{m} \cdot \left(2\epsilon + 2 \left(\frac{n}{d}\right)^{m-2} \cdot (\tau + \epsilon)\right) \\
\Leftrightarrow (d-1) &\geq 2 \frac{mn^2}{(m-2)d} \cdot \left(2\epsilon + 2 \left(\frac{n}{d}\right)^{m-2} \cdot (\tau + \epsilon)\right) \\
\Leftrightarrow (d-1) &\geq 2 \frac{mn^2}{(m-2)(d-1)} \cdot \left(2\epsilon + 2 \left(\frac{n}{d-1}\right)^{m-2} \cdot (\tau + \epsilon)\right) \\
\Leftrightarrow \left(2\epsilon + 2 \left(\frac{n}{d-1}\right)^{m-2} \cdot (\tau + \epsilon)\right) &\leq \frac{(m-2)(d-1)^2}{2mn^2} \\
\Leftarrow \left(2 \left(\frac{n}{d-1}\right)^{m-2} \cdot (\tau + 2\epsilon)\right) &\leq \frac{(m-2)(d-1)^2}{2mn^2} \\
\Leftrightarrow (\tau + 2\epsilon) &\leq \frac{m-2}{4m} \cdot \left(\frac{d-1}{n}\right)^m \\
\Leftarrow (\tau + 2\epsilon) &\leq \frac{1}{12} \cdot \left(\frac{d-1}{n}\right)^m
\end{aligned}$$

The above shows that the condition assumed in the lemma statement indeed is sufficient to establish a large independent set. Next we simplify the proximity bound obtained. We have

$$\begin{aligned}
\delta &\leq \tau + (1-\tau) \cdot \frac{2\gamma mn}{(m-2)d} \\
&\leq \tau + \frac{2\gamma mn}{(m-2)d} \\
&\leq \tau + \frac{2mn}{(m-2)d} \cdot \frac{m-1}{m} \cdot \left(2\epsilon + 2 \left(\frac{n}{d}\right)^{m-2} \cdot (\tau + \epsilon)\right) \\
&\leq \tau + \frac{2mn}{(m-2)d} \cdot \frac{m-1}{m} \cdot 2 \left(\frac{n}{d}\right)^{m-2} \cdot (\tau + 2\epsilon) \\
&= \tau + \frac{4(m-1)}{m-2} \cdot \left(\frac{n}{d}\right)^{m-1} \cdot (\tau + 2\epsilon) \\
&\leq \frac{4(m-1)}{m-2} \cdot \left(\frac{n}{d}\right)^{m-1} \cdot (2\tau + 2\epsilon) \\
&= \frac{8(m-1)}{m-2} \cdot \left(\frac{n}{d}\right)^{m-1} \cdot (\tau + \epsilon) \\
&\leq 16 \cdot \left(\frac{n}{d}\right)^{m-1} \cdot (\tau + \epsilon).
\end{aligned}$$

■

Next we improve the bound achieved on the proximity of  $r$  by looking at the structure of the graph  $G_m^n$  (the graph underlying the  $m$ -Product tester) and its “expansion”. Such improvements are a part of the standard toolkit in the analysis of low-degree tests based on axis parallel lines (see e.g., [7, 6, 16, 17] etc.) We follow the proof outline of [17] which in turn uses a proof technique of [10].

First, some notation: Fix  $n$  and  $m$  and the graph  $G_m^n$ . Let  $L$  and  $R$  denote the left and right vertices of  $G_m^n$ . Let  $d_L$  and  $d_R$  denote its left and right degrees. And let  $E$  denote the edges of  $G_m^n$ . Note  $|L| = n^m$ ,  $|R| = mn$ ,  $d_L = m$  and  $d_R = n^{m-1}$ . In particular,  $d_L \cdot |L| = d_R \cdot |R|$ . For a set  $A \subseteq L \cup R$ , let  $\Gamma(A) = \{(u, v) \in E \mid u \in A, v \notin A\}$ . Using this notation, we have the following:

**Lemma 3.4** *Fix  $n, m \geq 3$  and let  $L, R$  denote the two sides of the partition of the vertices of  $G_m^n$  and let  $d_L, d_R$  denote the left and right degrees. Let  $S \subseteq L$  and  $T \subseteq R$  be such that  $\frac{|S|}{|L|} \leq \frac{1}{4}$ . Then  $|\Gamma(S \cup T)| \geq \frac{d_L}{8} \cdot |S| + \frac{d_R}{8} \cdot |T|$ .*

**Proof:** We start with a simple observation that also allows us to bound the size of  $T$ . Suppose,  $|T| \geq |R|/2$ . Then the number of edges leaving  $T$  is at least  $d_R \cdot |T| \geq d_R \cdot (|R|/2)$ . On the other hand the number of edges entering  $S$  is at most  $d_L \cdot |S| \leq d_L \cdot (|L|/4)$ . Thus in this case, we have

$$\begin{aligned} \Gamma(S \cup T) &\geq d_R \cdot (|R|/2) - d_L \cdot (|L|/4) \\ &= d_R \cdot (|R|/4) \\ &= d_R \cdot (|R|/8) + d_L \cdot (|L|/8) \\ &\geq d_R \cdot (|S|/8) + d_L \cdot (|T|/8). \end{aligned}$$

We are thus reduced to the case where  $|S|/|L|, |T|/|R| \leq \frac{1}{2}$ . Here, we follow the proof of Babai and Szegedy [10]. (See also [21]). The crucial fact needed to apply their proof is that the graph  $G_m^n$  is edge-transitive, i.e., for every pair of edges  $e_1, e_2$  in  $G_m^n$ , there is an automorphism of  $G_m^n$  that maps  $e_1$  to  $e_2$ . This fact is used as follows: Let  $A$  denote the set of all automorphisms of  $G_m^n$ . Then if we consider any fixed edge  $e \in G_m^n$  and all its images under automorphisms  $A$  as a multiset, then every edge of  $G_m^n$  appears exactly the same number of times.

Armed with this fact, the proof proceeds as follows: For every pair  $u \in L$  and  $v \in R$  define a canonical shortest path  $P_{u,v}$ . Note that this path has length at most three. Note that an automorphism from  $A$  maps a path in  $G_m^n$  to a path in  $G_m^n$ . Now consider the multiset  $\mathcal{P}$  of all paths obtained by taking the paths  $P_{u,v}$  for every  $u, v$ , and their automorphisms for every automorphism in  $A$ . The cardinality of  $\mathcal{P}$  is thus  $|A| \cdot |L| \cdot |R|$ . The symmetry over the edges implies that every edge in  $E$  has exactly the same number, say  $N$ , of paths from  $\mathcal{P}$  passing through them. Since each path has at most three edges, we have  $N \leq \frac{3 \cdot |A| \cdot |R|}{d_L} = \frac{3 \cdot |A| \cdot |L|}{d_R}$ , or equivalently  $\frac{|A|}{N} \geq \frac{d_L}{3 \cdot |R|} = \frac{d_R}{3 \cdot |L|}$ .

Now consider the set of paths  $\mathcal{P}' \subseteq \mathcal{P}$  whose endpoints involve exactly one element of  $S \cup T$ . We have the cardinality of  $\mathcal{P}'$  equals  $|A| \cdot (|S| \cdot |\bar{T}| + |\bar{S}| \cdot |T|)$  (where  $\bar{S} = L - S$  and  $\bar{T} = R - T$ ). On the other hand, we have  $|\mathcal{P}'| \leq N \cdot |\Gamma(S \cup T)|$

Combining the two we have

$$\begin{aligned}
|\Gamma(S \cup T)| &\geq \frac{1}{N} \cdot |\mathcal{P}'| \\
&\geq \frac{|A|}{N} \cdot (|S| \cdot |\bar{T}| + |\bar{S}| \cdot |T|) \\
&\geq \frac{d_L}{3 \cdot |R|} \cdot |S| \cdot |\bar{T}| + \frac{d_R}{3 \cdot |L|} \cdot |\bar{S}| \cdot |T| \\
&\geq \frac{d_L}{6} \cdot |S| + \frac{d_R}{6} \cdot |T|.
\end{aligned}$$

This proves the lemma.  $\blacksquare$

**Lemma 3.5** *Let  $m$  be a positive integer and  $C$  be an  $[n, k, d]_\Sigma$  code with the property  $d^{m-1}/n^{m-1} \geq \frac{7}{8}$ . If  $r \in \Sigma^{n^m}$  and  $c \in C^m$  satisfy  $\delta(r, c) \leq \frac{1}{4}$  then  $\delta(r, c) \leq 8\rho(r)$ .*

**Proof:** Let  $L, R$  denote the two sides of the partition of the vertices of  $G_m^n$ . Note that the right vertices of  $G_m^n$  are of the form  $(b, i)$ , with  $b \in [m]$  and  $i \in [n]$ . Let  $r_{b,i}$  denote the projection of  $r$  to the neighborhood of the right vertex  $(b, i)$ , and let  $c_{b,i}$  denote the projection of  $c$  to the same. Let  $c'_{b,i}$  denote the codeword of  $C^{m-1}$  closest to  $r_{b,i}$ . Call an edge  $(u, (b, i))$  of  $G_m^n$  *bad* if  $r$  and  $c'_{b,i}$  disagree at  $u$ . Note that the fraction of bad edges equals  $\rho(r)$ .

We now lower bound  $\rho(r)$  in terms of  $\delta(r, c)$ . For this part we use Lemma 3.4. Let  $S \subseteq L$  be the set of vertices  $(i_1, \dots, i_m)$  for which  $r[i_1, \dots, i_m] \neq c[i_1, \dots, i_m]$ . Note that by assumption  $|S|/|L| = \delta(r, c) \leq \frac{1}{4}$ . Let  $T \subseteq R$  be the set of vertices  $(b, i)$  for whom  $c_{b,i} \neq c'_{b,i}$ . By Lemma 3.4 we have  $|\Gamma(S \cup T)| \geq \frac{d_L}{8} \cdot |S| + \frac{d_R}{8} \cdot |T|$ . We now claim that most of these edges are *bad*.

Consider first an edge  $(u, (b, i))$  in  $G_m^n$  from  $S$  to  $\bar{T}$ . On the one hand  $c'_{b,i} = c_{b,i}$  and on the other  $r[u] \neq c[u]$ . This leads to a disagreement between  $r$  and  $c'$  at  $u$  and so such an edge is bad. Next, consider an edge  $(u, (b, i))$  from  $u \in \bar{S}$  to  $T$ . We do have  $r[u] = c[u]$  and  $c'_{b,i} \neq c_{b,i}$ , but this doesn't imply that  $(u, (b, i))$  is bad, since  $c_{b,i}$  and  $c'_{b,i}$  need not disagree at  $u$ . Indeed for every  $(b, i) \in T$ , there may be up to  $n^{m-1} - d^{m-1}$  edges  $(u, (b, i))$  for which  $c'_{b,i}$  and  $r$  agree at  $u$ , but remaining edges out of  $(b, i)$  are bad. Discounting for these edges, we see that all but at most  $(n^{m-1} - d^{m-1}) \cdot |T|$  edges from  $T$  to  $\bar{S}$  are bad. Thus we get that the number of bad edges is at least  $\frac{d_L}{8} \cdot |S| + \frac{d_R}{8} \cdot |T| - (n^{m-1} - d^{m-1}) \cdot |T|$ . Using  $d_R = n^{m-1}$  and  $d^{m-1}/n^{m-1} \geq \frac{7}{8}$ , we get  $\frac{d_R}{8} \cdot |T| - (n^{m-1} - d^{m-1}) \cdot |T| \geq 0$ . Thus we get that the fraction of bad edges  $\beta$  is at least  $\frac{1}{8} \cdot (|S|/|L|) = \frac{\delta(r,c)}{8}$ . We conclude  $\delta(r, c) \leq 8 \cdot \rho(r)$ .  $\blacksquare$

We are now ready to put the pieces together to prove Theorem 2.6.

**Proof of Theorem 2.6:** Let  $\alpha = 2^{-14} \cdot \left(\frac{d-1}{n}\right)^{2m}$ . We will prove that the  $m$ -Product Tester is  $\alpha$ -robust for  $C^m$ . Note that  $\alpha \geq 2^{-16}$  as required for the theorem, and  $\sqrt{\alpha} \leq \min\left\{\frac{1}{36} \cdot \left(\frac{d-1}{n}\right)^m, \frac{1}{128} \cdot \left(\frac{d}{n}\right)^{m-1}\right\}$  (as will be required below).

The completeness (that codewords of  $C^m$  have expected robustness zero) follows from Proposition 2.5. For the soundness, consider any vector  $r \in \Sigma^{n^m}$  and let  $\rho = \rho(r)$ . If  $\rho > \alpha$ , then there is nothing to prove since  $\rho/\alpha > 1 \geq \delta_{C^m}(r)$ . So assume  $\rho \leq \alpha$ .

Note that  $r$  has  $\sqrt{\rho}$ -soundness-error at most  $\sqrt{\rho}$ . Furthermore, by the assumption on  $\rho$ , we have  $3\sqrt{\rho} \leq 3\sqrt{\alpha} \leq \frac{1}{12} \cdot \left(\frac{d-1}{n}\right)^m$  and so, by Lemma 3.3, we have  $\delta_{C^m}(r) \leq 16 \cdot \left(\frac{n}{d}\right)^{m-1} \cdot 2 \cdot \sqrt{\rho}$ . Now using  $\sqrt{\rho} \leq \sqrt{\alpha} \leq \frac{1}{128} \cdot \left(\frac{d}{n}\right)^{m-1}$ , we get  $\delta_{C^m}(r) \leq \frac{1}{4}$ . Let  $v$  be a codeword of  $C^m$  closest to  $r$ . We now have  $\delta(r, v) \leq \frac{1}{4}$  and  $\left(\frac{d}{n}\right)^{m-1} \geq \frac{7}{8}$ , and so, by Lemma 3.5, we get  $\delta_{C^m}(r) = \delta(r, v) \leq 8\rho$ . This concludes the proof.  $\blacksquare$

## 4 Tanner Product Codes and Composition

In this section we define the composition of two Tanner Product Codes, and show how they preserve robustness. We then use this composition to show how to test  $C^m$  using projections to  $C^2$ .

### 4.1 Composition

Recall that a Tanner Product Code is given by a pair  $(G, C_{\text{small}})$ . We start by defining a composition of graphs that corresponds to the composition of codes.

Given an  $(N, M, D)$ -ordered graph  $G = \{\ell_1, \dots, \ell_M\}$  and an additional  $(D, m, d)$ -ordered graph  $G' = \{\ell'_1, \dots, \ell'_m\}$ , their Tanner Composition, denoted  $G \odot G'$ , is an  $(N, M \cdot m, d)$ -ordered graph with adjacency lists  $\{\ell''_{j,j'} | j \in [M], j' \in [m]\}$ , where  $\ell''_{(j,j'),i} = \ell_{j,\ell'_{j',i}}$ .

**Lemma 4.1 (Composition)** *Let  $G_1$  be an  $(N, M, D)$ -ordered graph, and  $C_1 \subseteq \Sigma^D$  be a linear code with  $C = \text{TPC}(G_1, C_1)$ . Further, let  $G_2$  be an  $(D, m, d)$ -ordered graph and  $C_2 \subseteq \Sigma^d$  be a linear code such that  $C_1 = \text{TPC}(G_2, C_2)$ . Then  $C = \text{TPC}(G_1 \odot G_2, C_2)$  (giving a  $d$ -query local test for  $C$ ). Furthermore if  $(G_1, C_1)$  is  $c_1$ -robust and  $(G_2, C_2)$  is  $c_2$ -robust, then  $(G_1 \odot G_2, C_2)$  is  $c_1 \cdot c_2$ -robust.*

**Proof:** We focus on the robustness of the  $C$ , as all other claims follow immediately from definitions. Assume  $w \in \Sigma^N$  has distance  $\delta$  from  $C$ . Then, since  $C = \text{TPC}(G_1, C_1)$  is  $c_1$ -robust, the expected distance of a random "medium"-size test (of query size  $D$ ) is at least  $\delta c_1$ , so by the  $c_2$ -robustness of  $C_1 = \text{TPC}(G_2, C_2)$  the expected distance of the "small"-size test (of query complexity  $d$ ) is at least  $\delta c_1 \cdot c_2$  as claimed.

$\blacksquare$

### 4.2 Testing a 4-Wise Tensor Product Code

We continue by recasting the results of Section 3 in terms of robustness of associated Tanner Products. Recall that  $G_m^n$  denotes the graph that corresponds to the tests of  $C^m$  by the  $m$ -Product Tester, where  $C \subseteq \Sigma^n$ .

Note that  $G_m^n$  can be composed with  $G_{m-1}^n$  and so on. For  $m' < m$ , define  $G_{m,m'}^n = G_m^n$  if  $m' = m-1$  and define  $G_{m,m'}^n = G_m^n \odot G_{m-1,m'}^n$  otherwise. Thus we have that  $C^m = \text{TPC}(G_{m,m'}^n, C^{m'})$ . The following lemma (which follows easily from Theorem 2.6 and Lemma 4.1) gives the robustness of  $(G_{4,2}^n, C^2)$ .

**Lemma 4.2** *Let  $C$  be an  $[n, k, d]_\Sigma$  code with  $((d-1)/n)^4 \geq \frac{7}{8}$ . Then  $(G_{4,2}^n, C^2)$  is  $2^{-32}$ -robust.*

**Proof:** Since we have  $((d-1)/n)^4 \geq \frac{7}{8}$  we may apply Theorem 2.6 with  $m = 3, 4$  to get that  $(G_4^n, C^3)$  and  $(G_3^n, C^2)$  are both  $2^{-16}$ -robust. Since  $C^3 = \text{TPC}(G_3^n, C^2)$ , we may apply Lemma 4.1 to conclude that  $(G_{4,2}^n = G_4^n \odot G_3^n, C^2)$  is  $2^{-32}$ -robust.  $\blacksquare$

### 4.3 Testing Tensor Products with $C^2$ tests

Finally we define graphs  $H_t^n$  so that  $C^{2^t} = \text{TPC}(H_t^n, C^2)$ . This is done recursively by letting  $H_2^n = G_{4,2}^n$  and letting  $H_t^n = G_{4,2}^{n^{2^{t-2}}} \odot H_{t-1}^n$  for  $t > 2$ . We now analyze the robustness of  $(H_t^n, C^2)$ .

**Lemma 4.3** *There exists a constant  $\alpha > 0$  such that the following holds: Let  $t$  be an integer and  $C$  be an  $[n, k, d]_\Sigma$  code such that  $d-1 \geq (1 - \frac{1}{5m}) \cdot n$ , for  $m = 2^t$ . Then  $(H_t^n, C^2)$  is  $\alpha^t$ -robust.*

**Proof:** Note that the condition in the lemma implies  $((d-1)/n)^m \geq (1 - \frac{1}{5m})^m \geq e^{-0.1} \geq \frac{7}{8}$ . This is the form in which we use the condition.

We prove the lemma, for  $\alpha = 2^{-32}$ , by induction. For the base case, we have  $(H_2^n = G_{4,2}^n, C^2)$  is  $2^{-32}$ -robust, by Lemma 4.2. (Here we use the fact that  $(d-1)/n \geq \frac{7}{8}$  as needed.)

For the induction, let  $m = 2^t$ . and let  $C' = C^{m/4}$ . Let  $G_1 = G^{m/4}$ ,  $C_1 = (C')^2$ ,  $G_2 = H_{t-1}^n$  and  $C_2 = C^2$ . Note that  $H_t^n = G_1 \odot G_2$  and  $C_1 = \text{TPC}(G_2, C_2)$ . Thus we can bound the robustness of  $(H_t^n, C_2)$  by bounding the robustness of  $(G_1, C_1)$  and  $(G_2, C_2)$  and then using Lemma 4.1. Note that  $C_1 = (C')^2$  and  $C'$  is a  $[n^{m/4}, k^{m/4}, d^{m/4}]_\Sigma$  code, where

$$\left( \frac{d^{m/4} - 1}{n^{m/4}} \right)^4 \geq \left( \frac{d-1}{n} \right)^m \geq \frac{7}{8}.$$

Thus we can apply Lemma 4.2 to conclude  $(G_1, C_1) = (G_{4,2}^n, (C')^2)$  is  $\alpha$ -robust for  $\alpha = 2^{-32}$ . By induction, we also have  $(G_2, C_2) = (H_{t-1}^n, C^2)$  is  $\alpha^{t-1}$ -robust. By Lemma 4.1,  $(G_1 \odot G_2, C_2)$  is  $\alpha^t$ -robust.  $\blacksquare$

We are ready to prove Theorem 2.7.

**Proof of Theorem 2.7:** Let  $\alpha$  be the constant given by Lemma 4.3. Fix  $i$  and let  $C = C_i$ ,  $n = n_i$  etc. (i.e., we suppress the subscript  $i$  below). Then  $C^m$  is an  $[N, K, D]_q$  code, for  $N = n^m$ ,  $K = k^m$  and  $D = d^m$ . Since  $d/n \geq 1 - \frac{1}{2m}$ , we have  $C^m$  has relative distance  $d^m/n^m \geq \frac{1}{2}$ . Regarding query complexity, we have  $C^m = \text{TPC}(H_{\log_2 m}^n, C^2)$ , where  $(H_{\log_2 m}^n, C^2)$  is an  $\alpha^{\log_2 m}$ -robust tester for  $C^m$  and this tester has query complexity  $O(n^2)$ . From Proposition 2.3 we get that there is a tester for  $C$  that makes  $O(n^2/\alpha^{O(\log_2 m)}) = O(n^2 m^{O(1)})$  queries.

In particular, if  $n \leq p(k)$  then the rate of the code is inverse polynomial, i.e.,  $N = n^m = (p(k))^m \leq \text{poly}(k^m) = \text{poly}(K)$ . Similarly, if  $n \leq \hat{p}(m)$  then the query complexity is  $\text{poly}(n, m)$  which is  $\text{poly}(\log N)$ .  $\blacksquare$

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