Geometric Pattern Matching

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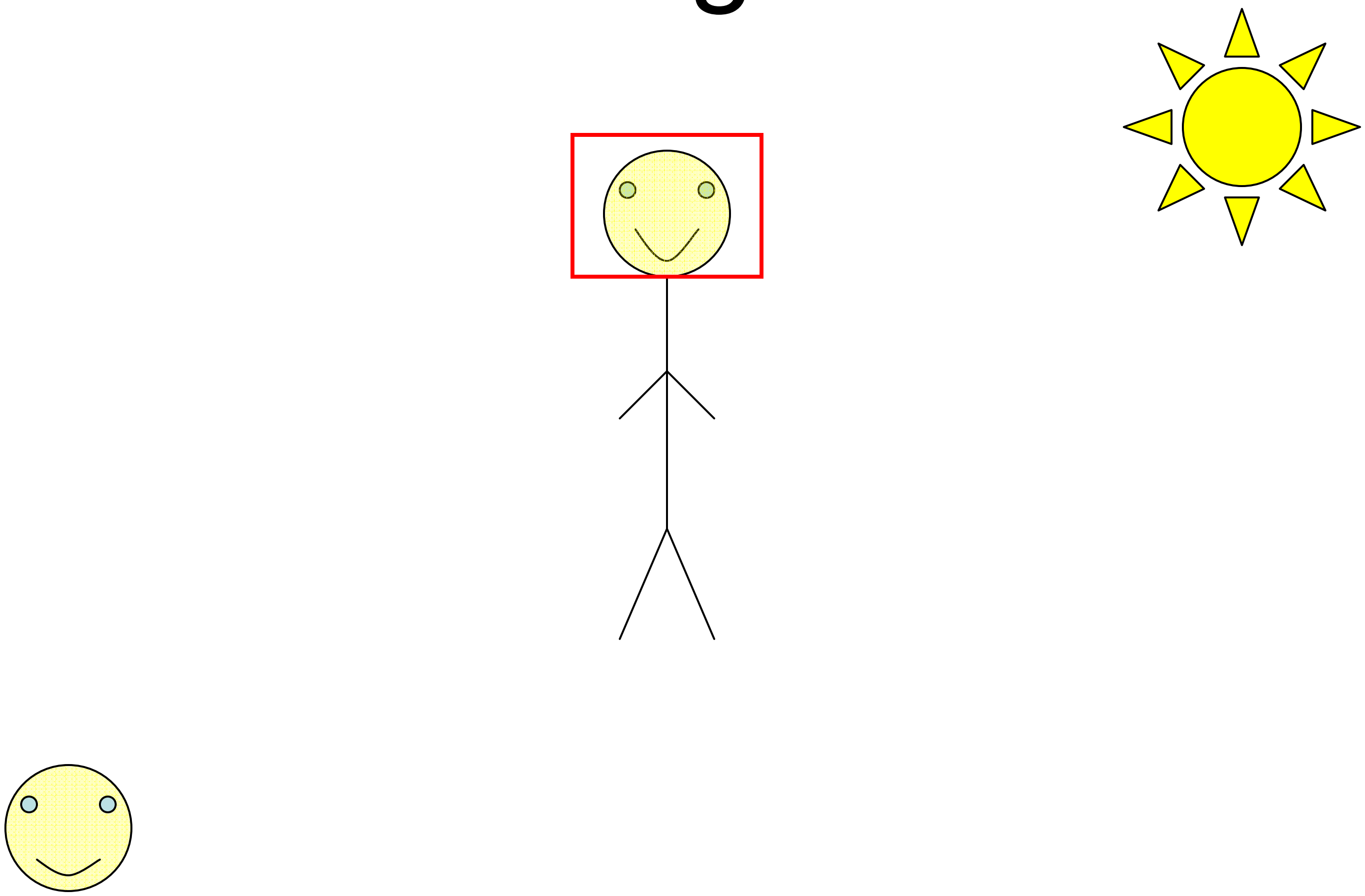
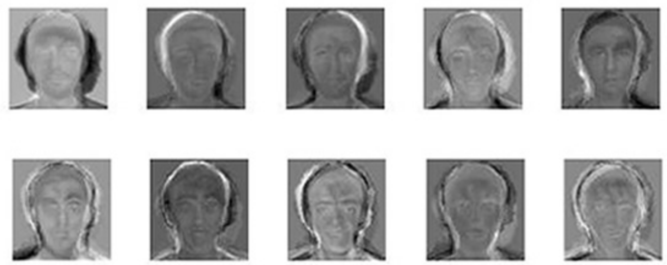
Matching in a Scene



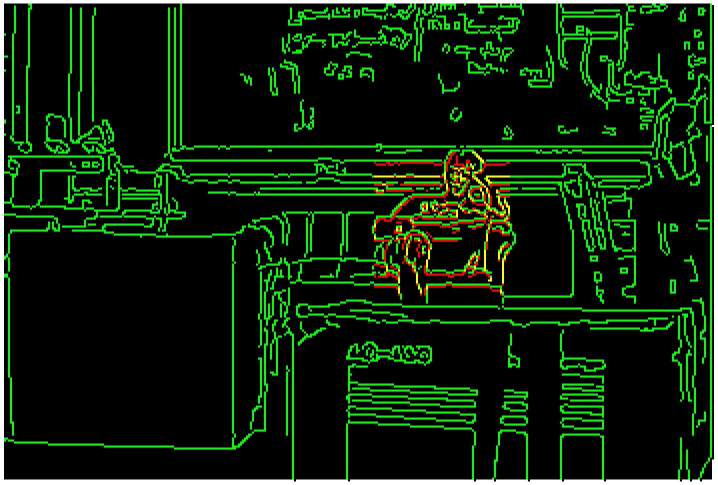
Eigenfaces

= 0.02 + 0.03   +0.01 +………..



Formalization: Shapes

- Today:
  - A shape is a set A of points in R^2
  - $|A|=n$
- In general, A could consist of segments etc.



Formalization: (Dis)similarity

- Hausdorff distance
  - $DH(A,B)=\max_{a\in A} \min_{b\in B} ||a-b||$
  - $H(A,B)=\max[\ DH(A,B), DH(B,A)\ ]$
- Earth-Mover Distance
  - Minimum cost of a one-to-one matching between A and B
  - $EMD(A,B)=\min_{f:A\rightarrow B,} \sum_{a\in A} ||a-f(a)||$, f is 1:1
- Bottleneck matching
  - $BM(A,B)= \min_{f:A\rightarrow B,} \max_{a\in A} ||a-f(a)||$, f is 1:1

Alignment

- In general, A and B are not **aligned**
- So, in general, we want
  - $DH_T(A,B)=\min_{t\in T} DH(t(A),B)$, where
    - T=translations
    - T=translations and rotations
    - …
  - Same for H

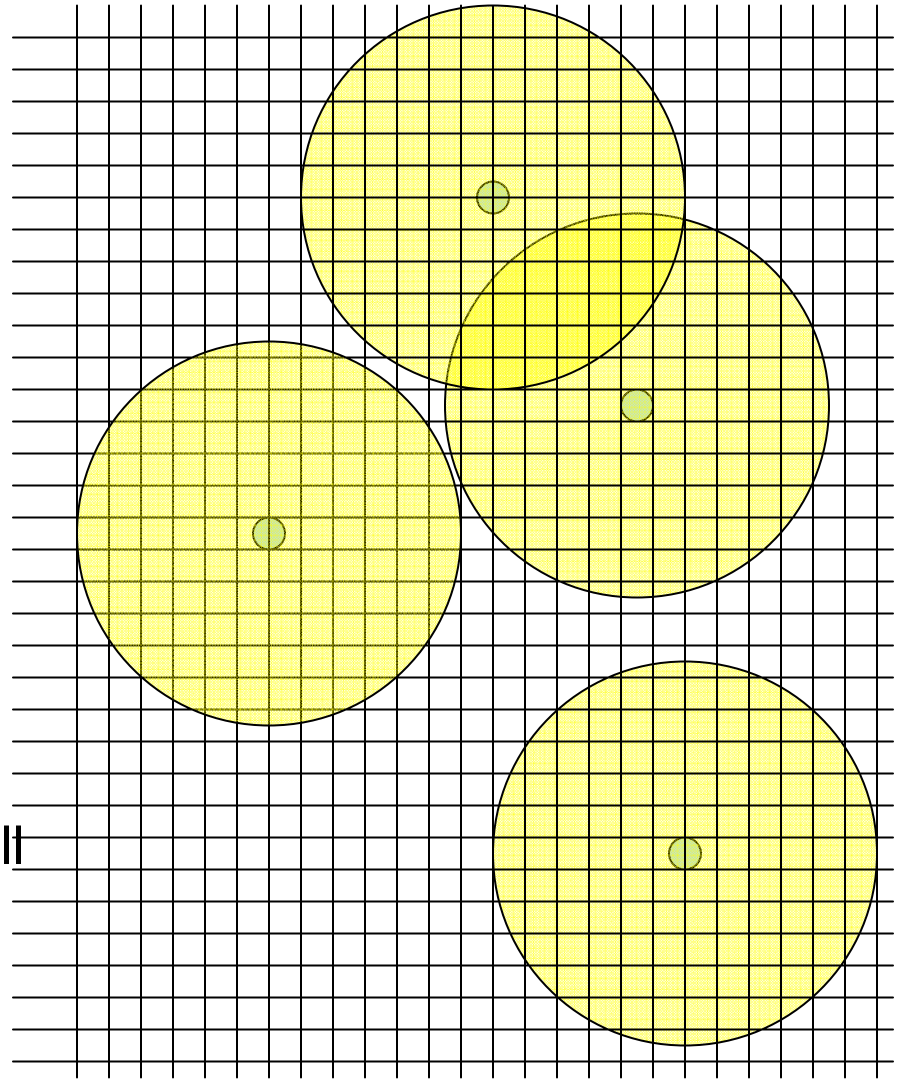
Algorithms

• Exact DH	O(n log n)
• Approx. DH	O(n)
• Exact DH_T	$O(n^4)$
• Approx. DH_T	$O(n^2)$
• Approx. H_T	O(n log n)
• $DH_{T+R}=0$	O()

Computing H(A,B)

- Given A,B, how fast can we compute H(A,B) ?
  - We will compute DH(A,B)
  - Construct a Voronoi diagram V for B
  - Construct a point location structure for V
  - For each a in A, find its NN in B
- Total time: O(n log n)

Approximate Hausdorff



- Assume we just want an algorithm that:
  - If $DH(A,B) \leq r$, answers YES
  - If $DH(A,B) \geq (1+\varepsilon)r$, answers NO
- Algorithm:
  - Impose a grid with cell diameter εr
  - For each $b \in B$, mark all cells within distance r from b
  - For each $a \in A$, check if a's cell is marked
- Time: $O(n/\varepsilon^2)$

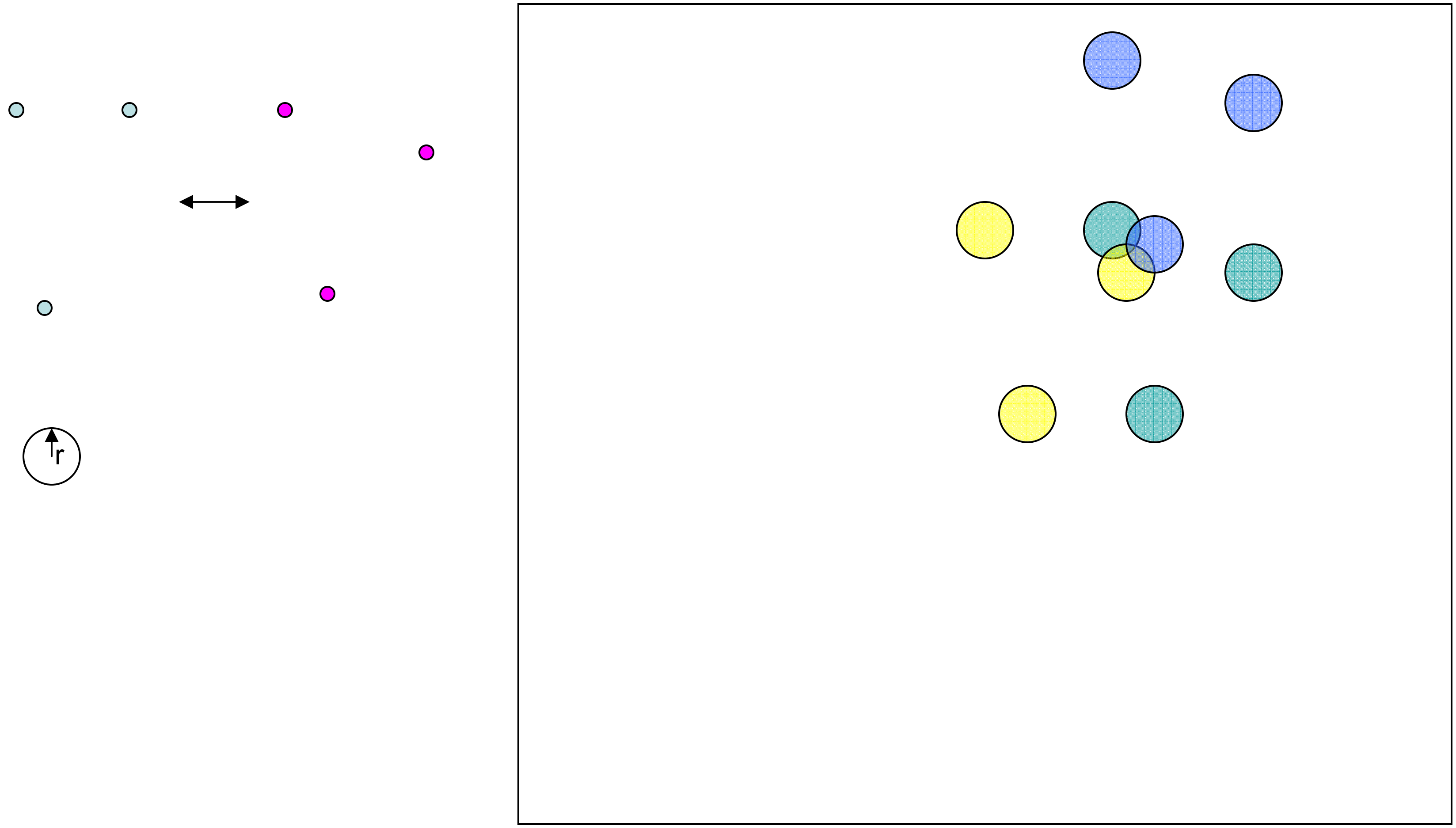
Decision Problem

- Again, focus on if $DH_T(A,B) \leq r$
- For $a \in A$, define

$$T(a)=\{ t: \exists b \in B \ ||t(a)-b|| \leq r \}$$

- $DH_T(A,B) \leq r$ iff $\cap_{a \in A} T(a)$ is non-empty

Example



Algorithm

- Compute the arrangement of all disks
- Check for non-emptiness
- Analysis:
  - Number of disks: n^2
  - Size of the arrangement: $O(n^4)$
  - Can compute the arrangement in time $O(n^4)$
  - Total time: $O(n^4)$
- Can be improved to $O(n^3 \log n)$ [Chew,Goodrich,Huttenlocher,Kedem,Kleinberg, Kravets'93]

Running time

- Running time pretty high
- Can we speed it up?
- A $(1+\varepsilon)$-approximate algorithm for the decision version of DH_T:
  - Impose a grid will pixel diameter εr
  - Approximate each disk by $O(1/\varepsilon^2)$ cells
  - Each $T(a)$ is approximated by $O(n/\varepsilon^2)$ cells
  - Need to check intersection of n $T(a)$'s
  - $O(n^2/\varepsilon^2)$ time

Approximate Algorithm for H()

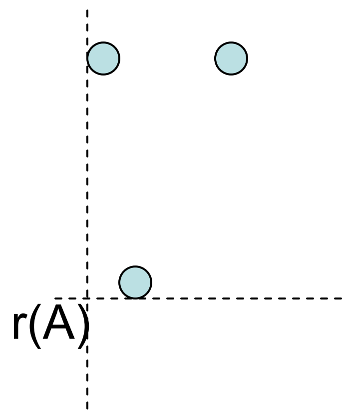
- Reference point: a point r(A) such that if we set t=r(B)-r(A) , then

$$H(\ t(A),\ B\ ) \leq c\ H_T(A,B)$$

- Note that t transforms r(A) into r(B)
- This will give us a c-approximate algorithm with time O(n log n)

Constructing a Ref Point

- $r(A)=(\min_{a \in A} a_x, \min_{a \in A} a_y)$



- Introduced in [Alt, Behrends, Blomer'91]
- Improved in [Aichholzer, Alt, Rote'94]

Theorem

Theorem: $r(A)$ is a reference point with $c=1+\sqrt{2}$

Proof:

- Consider optimal t ,i.e., s.t. $H(t(A),B)=H_T(A,B)$
- What is $|\min_{a\in A} t(a)_x - \min_{b\in B} b_x|$?
- It is at most $H_T(A,B)$
- Same for $|\min_{a\in A} t(a)_y - \min_{b\in B} b_y|$
- Thus $||r(t(A))-r(B)|| \leq \sqrt{2}\, H_T(A,B)$
- We can translate $t(A)$ by a vector of length $\sqrt{2}\, H_T(A,B)$, so that $r(A)$ is aligned with $r(B)$

Exact matching under T+R

- Check if $DH_{T+R}(A,B)=0$
  - Can modify the algorithm to allow small error
  - Technique used in practice

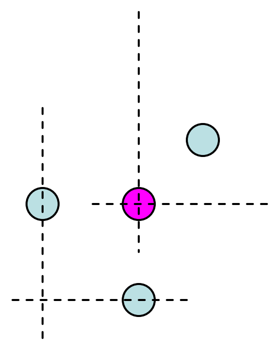
Algorithm

- Take any pair $a,a' \in A$, let $r=||a-a'||$
- Find all pairs $b,b' \in B$ such that $||b-b'||=r$
- For all such pairs
  - Compute t that transforms (a,a') into (b,b')
  - Check if $t(A) \subseteq B$

Analysis

- Choosing a,a' : constant time
- Enumerating b,b' : $O(n^2)$ time
- Checking match: $O(n P)$, where P is the number of pairs $b,b' \in B$ s.t. $||b-b'||=r$
- The bound for P is …
- …deferred to the next episode ☺

The $(1+\sqrt{2})$ Factor is Tight (for this reference point)



Face Recognition