# Segment Intersection

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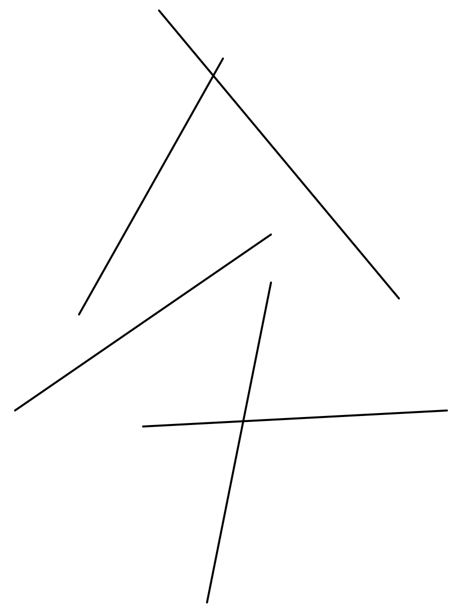
# Computational Geometry ctd.

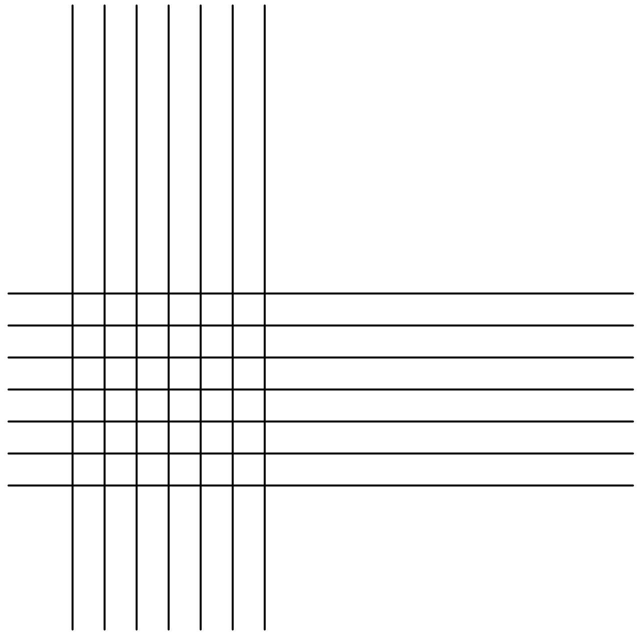
- Segment intersection problem:
  - Given: a set of $n$ distinct segments $s_1 \ldots s_n$, represented by coordinates of endpoints
  - Detection: detect if there is any pair $s_i \neq s_j$ that intersects
  - Reporting: report all pairs of intersecting segments



# Segment intersection

- Easy to solve in $O(n^2)$ time
- Is it possible to get a better algorithm for the reporting problem ?
- NO (in the worst-case)
- However:
  - We will see we can do better for the detection problem
  - Moreover, the number of intersections P is usually small.

    Then, we would like an *output sensitive* algorithm, whose running time is low if P is small.

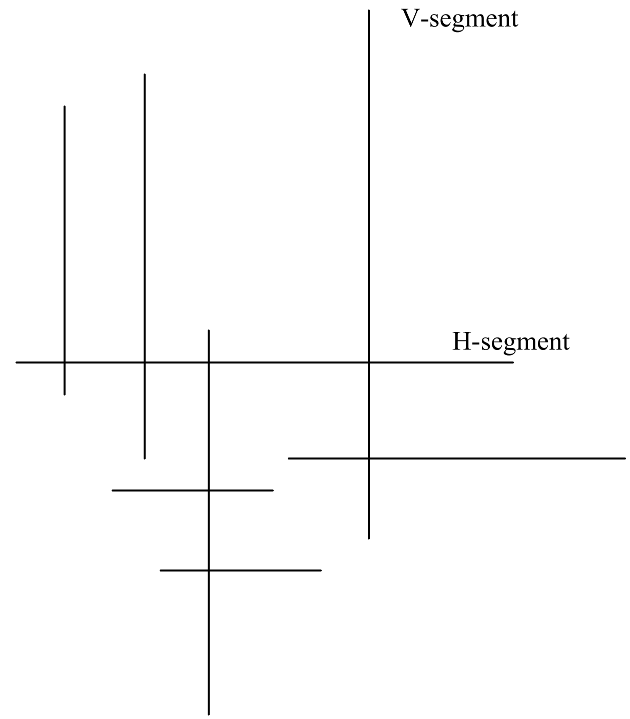


# Result

- We will show:
  - O(n log n) time for detection
  - O( (n +P) log n) time for reporting
- We will use *line sweep approach*
  - Many other applications, e.g.,Voronoi diagrams, motion planning

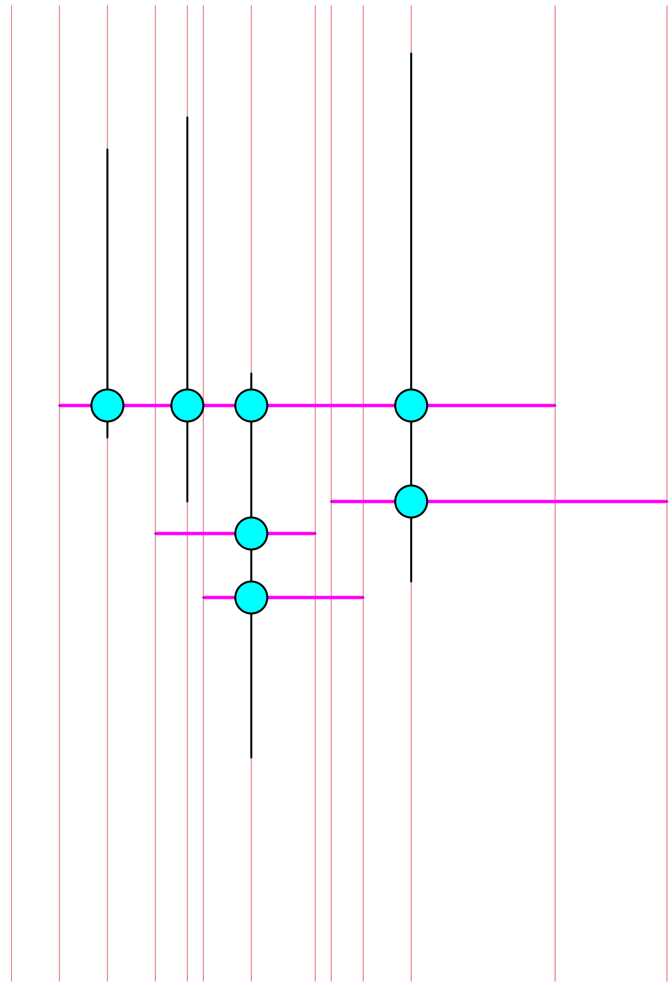
# Orthogonal segments

- All segments are either horizontal or vertical
- Assumption: all coordinates are distinct
- Therefore, only vertical-horizontal intersections exist



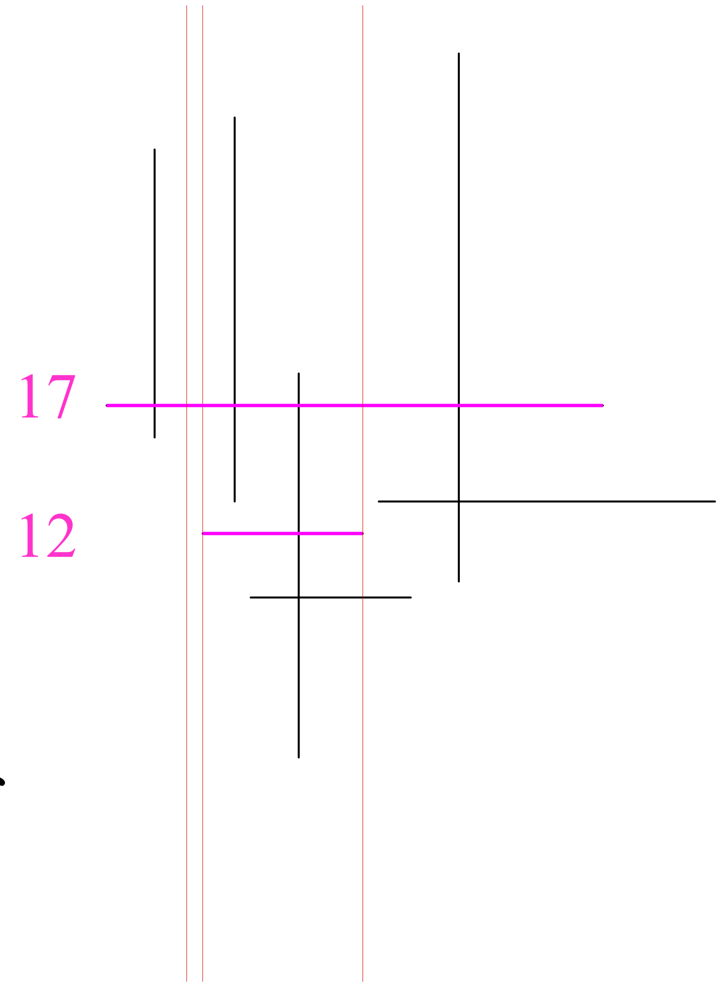
# Orthogonal segments

- Sweep line:
  - A *vertical line* sweeps the plane from left to right
  - It “stops” at all “important” x-coordinates, i.e., when it hits a V-segment or endpoints of an H-segment
  - Invariant: all intersections on the left side of the sweep line have been already reported



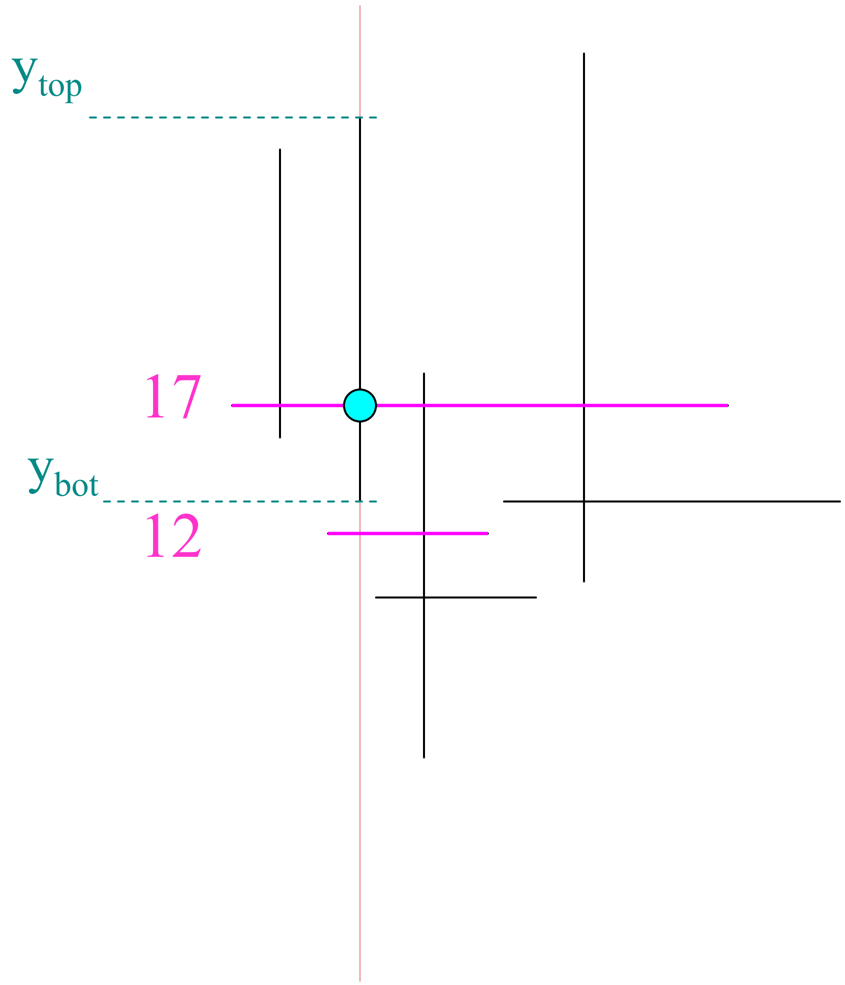
# Orthogonal segments ctd.

- We maintain sorted y-coordinates of H-segments currently intersected by the sweep line (using a balanced BST V)
- When we hit the left point of an H-segment, we add its y-coordinate to V
- When we hit the right point of an H-segment, we delete its y-coordinate from V



# Orthogonal segments ctd.

- Whenever we hit a V-segment having coord. $y_{top}$, $y_{bot}$), we report all H-segments in V with y-coordinates in $[y_{top}, y_{bot}]$



# Algorithm

- Sort all V-segments and endpoints of H-segments by their x-coordinates – this gives the "trajectory" of the sweep line
- Scan the elements in the sorted list:
  - Left endpoint: add segment to tree V
  - Right endpoint: remove segment from V
  - V-segment: report intersections with the H-segments stored in V

# Analysis

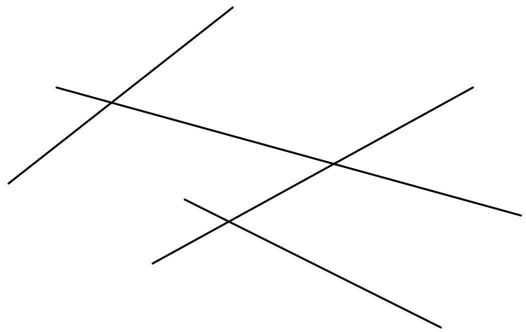
- Sorting: O(n log n)
- Add/delete H-segments to/from vertical data structure V:
  - O(log n) per operation
  - O(n log n) total
- Processing V-segments:
  - O(log n) per intersection - SEE NEXT SLIDE
  - O(P log n) total
- Overall: O( (P+ n) log n) time
- Can be improved to O(P +n log n)

# Analyzing intersections

- Given:
  - A BST $V$ containing $y$-coordinates
  - An interval $I=[y_{bot},y_{top}]$
- Goal: report all $y$'s in $V$ that belong to $I$
- Algorithm:
  - $y$=Successor($y_{bot}$)
  - While $y \le y_{top}$
    - Report $y$
    - $y$:=Successor($y$)
  - End
- Time: (number of reported $y$'s)*$O(\log n) + O(\log n)$

# The general case

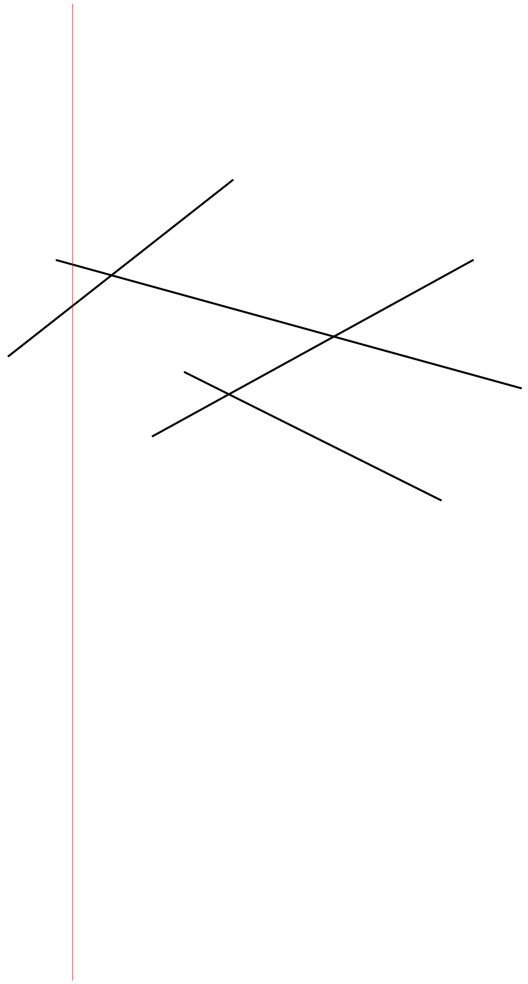
- Assumption: all coordinates of endpoints and intersections distinct
- In particular:
  - No vertical segments
  - No three segments intersect at one point



# Sweep line

- Invariant (as before): all intersections on the left of the sweep line have been already reported
- Stops at all "important" x-coordinates, i.e., when it hits endpoints or intersections
- Problem I: Do not know the intersections in advance !
- The list of intersection coordinates is constructed and maintained *dynamically*

  (in a "horizontal" data structure H)

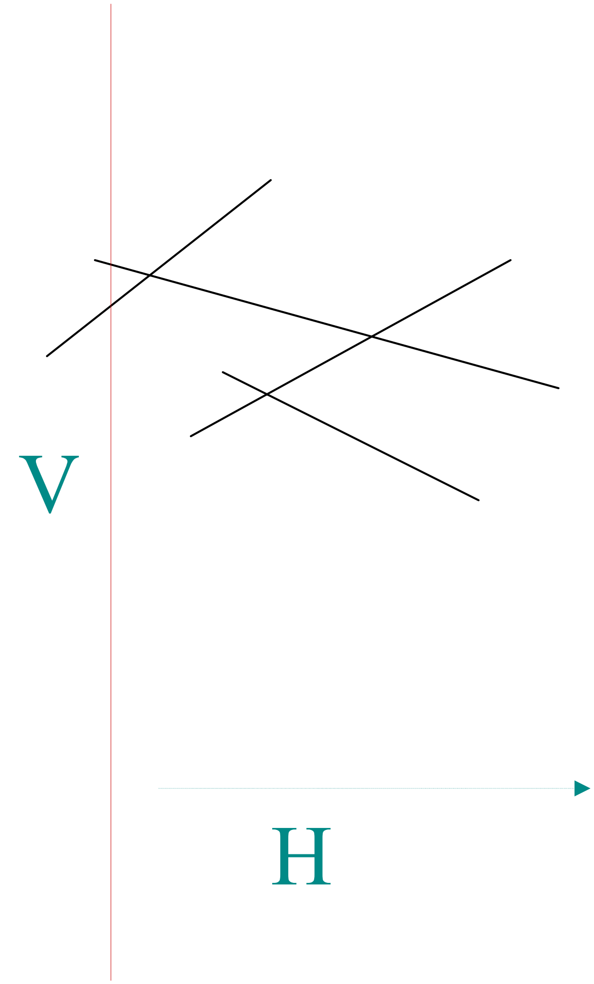


# Sweep line

- Also need to maintain the information about the segments intersecting the sweep line
- Problem II: Cannot keep the values of y-coordinates of the segments !
- Instead, we will maintain their *order* .I.e., at any point, we maintain all segments intersecting the sweep line, sorted by the y-coordinates of the intersections

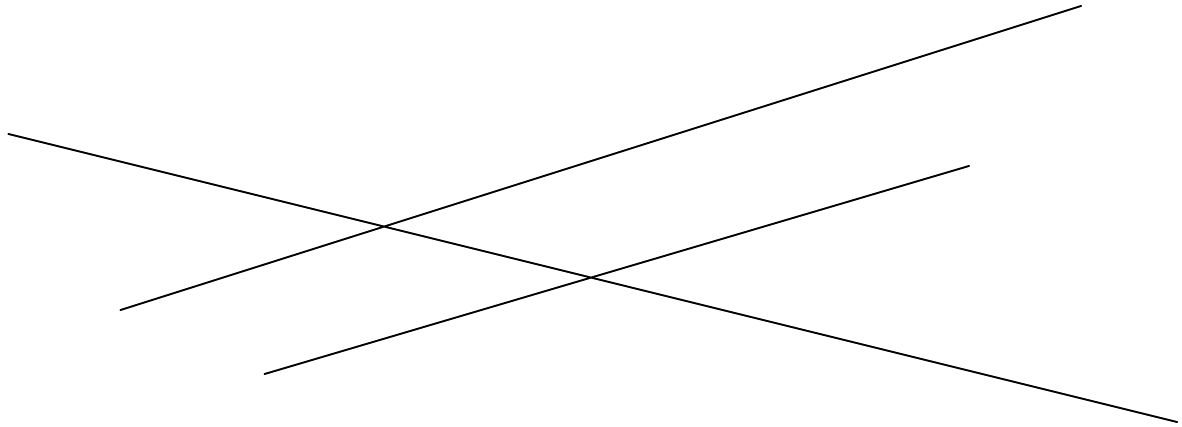
(in a "vertical" data structure V)

- Key idea: check only for the intersections of segments that are neighbors in V



# Java applet

- http://www.lems.brown.edu/~wq/projects/cs252.html
- Example used in the lecture:



# Algorithm

- Initialize the "vertical" BST V (to "empty")
- Initialize the "horizontal" priority queue H (to contain the segments' endpoints sorted by x-coordinates)
- Repeat
  - Take the next "event" p from H:

  // Update V
  - If p is the left endpoint of a segment, add the segment to V
  - If p is the right endpoint of a segment, remove the segment from V
  - If p is the intersection point of s and s', swap the order of s and s' in V, report p

(continued on the next slide)

# Algorithm ctd.

// Update H

– For each new pair of neighbors s and s’ in V:
  - Check if s and s’ intersect on the right side of the sweep line
  - If so, add their intersection point to H
  - Remove the possible duplicates in H

- Until H is empty

# Analysis

- Initializing H: O(n log n)
- Updating V:
  - O(log n) per operation
  - O( (P+n) log n) total
- Updating H:
  - O(log n) per intersection
  - O(P log n) total
- Overall: O( (P+ n) log n) time

# Correctness

- All reported intersections are correct
- Assume there is an intersection not reported. Let p=(x,y) be the first such unreported intersection (of s and s' )
- Let x' be the last event before p. Observe that:
  - At time x' segments s and s' are neighbors on the sweep line
  - Since no intersections were missed till then, V maintained the right order of intersecting segments
  - Thus, s and s' were neighbors in V at time x'. Thus, their intersection should have been detected

# Next week

- Randomized algorithm for linear programming
- Very efficient in low dimensions