Closest Pair

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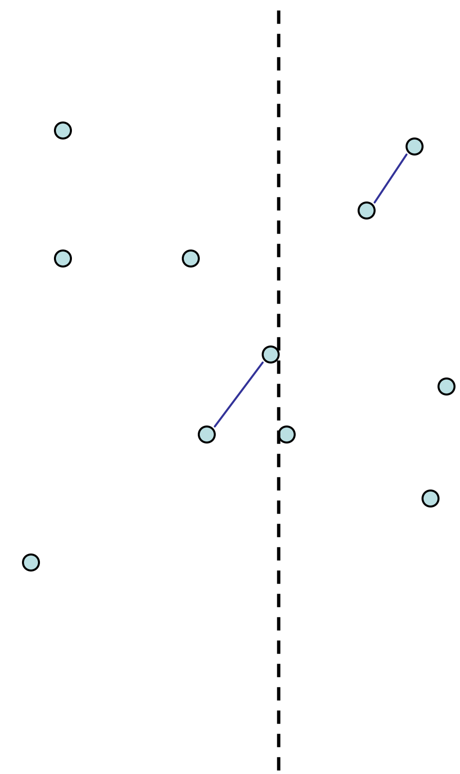
Closest Pair

- Find a closest pair among $p_1 \ldots p_n \in R^d$
- Easy to do in $O(dn^2)$ time
  - For all $p_i \neq p_j$, compute $||p_i - p_j||$ and choose the minimum
- We will aim for better time, as long as d is "small"
- For now, focus on $d=2$

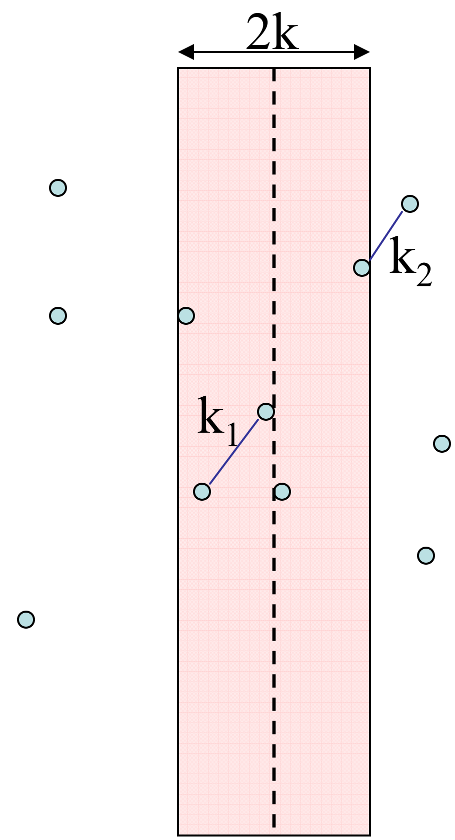
Divide and conquer

- Divide:
  - Compute the median of x-coordinates
  - Split the points into P_L and P_R, each of size $n/2$
- Conquer: compute the closest pairs for P_L and P_R
- Combine the results (the hard part)

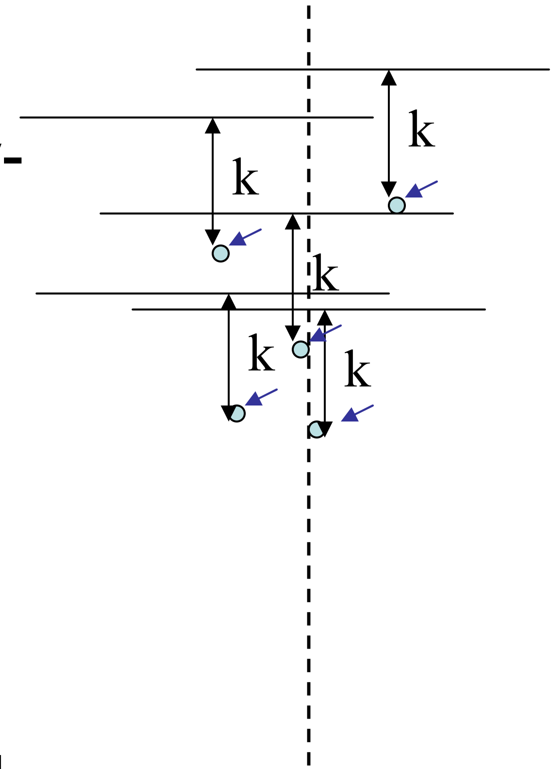


Combine

- Let $k=\min(k_1,k_2)$
- Observe:
  - Need to check only pairs which cross the dividing line
  - Only interested in pairs within distance $< k$
- Suffices to look at points in the $2k$-width strip around the median line



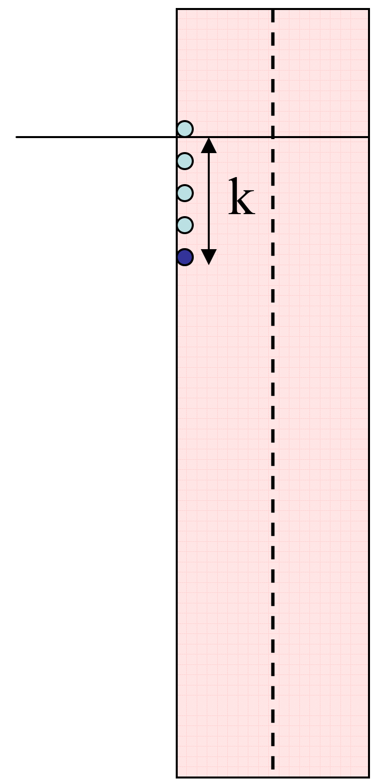
Scanning the strip



- Sort all points in the strip by their y-coordinates, forming $q_1 \ldots q_t$, $t \leq n$.
- Let y_i be the y-coordinate of q_i
- $k_{min} = k$
- For $i=1$ to t
  - $j=i-1$
  - While $y_i - y_j < k$
    - If $\|q_i - q_j\| < k_{min}$ then $k_{min} = \|q_i - q_j\|$
    - $j:=j-1$
- Report k_{min} (and the corresponding pair)

Analysis

- Correctness: easy
- Running time is more involved
- Can we have many q_j's that are within distance k from q_i ?
- No
- Proof by *packing* argument

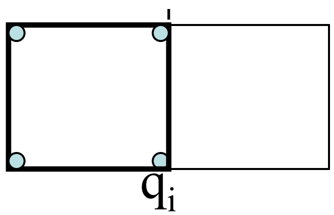


Analysis, ctd.

Theorem: there are at most 7 q_j's , $j<i$, such that $y_i - y_j \le k$.

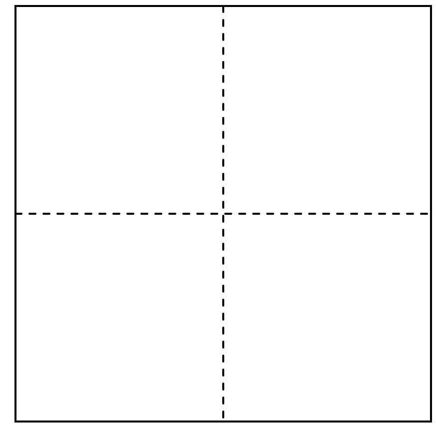
Proof:

- Each such q_j must lie either in the left or in the right $k \times k$ square
- Within each square, all points have distance distance $\ge k$ from others
- We can pack at most 4 such points into one square, so we have 8 points total (incl. q_i)



At most 4

- Split the square into 4 sub-squares of size $k/2 \times k/2$
- Diameter of each square is $k/2^{1/2} < k \rightarrow$ at most one point per sub-square



Running time

- Divide: $O(n)$
- Combine: $O(n \log n)$ because we sort by y
- However, we can:
  - Sort all points by y at the beginning
  - Divide preserves the y-order of points

  Then combine takes only $O(n)$
- We get $T(n)=2T(n/2)+O(n)$, so $T(n)=O(n \log n)$

Higher dimensions (sketch)

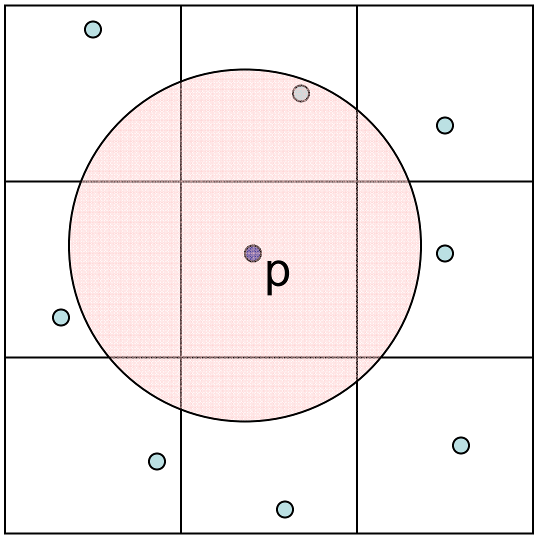
- Divide: split P into P_L and P_R using the hyperplane $x=t$
- Conquer: as before
- Combine:
  - Need to take care of points with x in $[t-k,t+k]$
  - This is essentially the same problem, but in $d-1$ dimensions
  - We get:
    - $T(n,d)=2T(n/2, d)+T(n,d-1)$
    - $T(n,1)=O_d(1)\ n$
  - Solves to: $T(n,d)=n \log^{d-1} n$

Closest Pair with Help

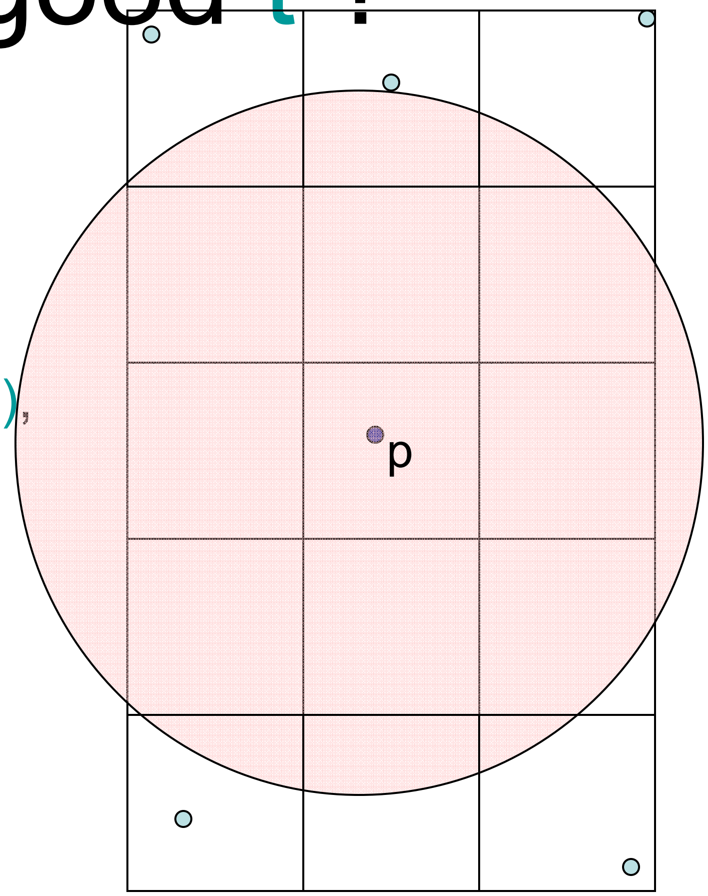
- Given: $P=\{p_1 \ldots p_n\}$ of points from R^d, such that the closest distance is in $(t,c\ t]$
- Goal: find the closest pair
- Will give an $O((2c\ d^{1/2})^d\ n)$ time algorithm
- Note: by scaling we can assume $t=1$

Algorithm

- Impose a cubic grid onto R^d, where each cell is a $1/d^{1/2} \times 1/d^{1/2}$ cube
- Put each point into a bucket corresponding to the cell it belongs to
- Diameter of each cell is ≤ 1, so at most one point per cell
- For each $p \in P$, check all points in cells intersecting a ball $B(p,c)$
- How many cells are there ?
  - All are contained in a d-dimensional box of side $2(c+1/d^{1/2}) \leq 2(c+1)$
  - At most $(2\, d^{1/2}\ (c+1)\ )^d$ such cells
- Total time: $O((2c\, d^{1/2})^d\, n)$



How to find good t ?



- Repeat:
  - Choose a random point p in P
  - Let t=t(p)=D(p,P-{p})
  - Impose a grid with side $t' < t/(1+d^{1/2})$, i.e., such that any pair of adjacent cells has diameter <t
  - Put the points into the grid cells
  - Remove all points whose all adjacent cells are empty
- Until P is empty
- Observation: the values t are decreasing

Correctness

- Consider t computed in the last iteration
  - There is a pair of points with distance t (it defines t)
  - There is no pair of points with distance t' or less (otherwise they would have been placed in adjacent cells, and the algorithm would have continued)
  - We get $c=t/t' \sim 2\, d^{1/2}$

Running time

- Consider $t(p_1) \ldots t(p_m)$
- An iteration is lucky if $t(p_i) \geq t$ for at last half of points p_i
- The probability of being lucky is $\geq 1/2$
- Expected #iterations till a lucky one is ≤ 2
- After we are lucky, the number of points is $\leq m/2$
- Total expected time = 3^d times $O(n+n/2+n/4+\ldots+1)$