Delaunay Triangulations

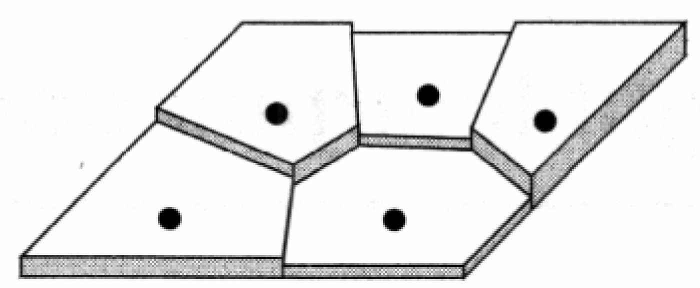
(slides mostly by Glenn Eguchi)

Motivation: Terrains

- Set of data points $A \subset R^2$
- Height $f(p)$ defined at each point p in A
- How can we most naturally approximate height of points not in A?

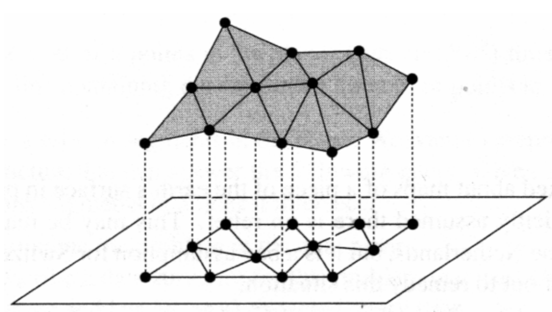
Option: Discretize

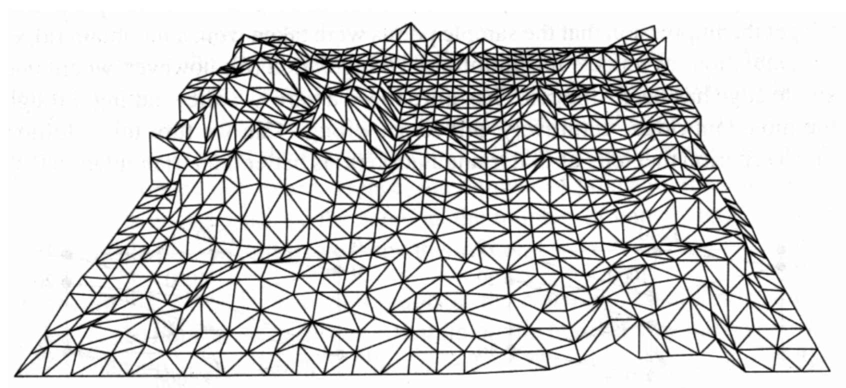
- Let $f(p)$ = height of nearest point for points not in A
- Does not look natural



Better Option: Triangulation

- Determine a *triangulation* of A in R^2, then raise points to desired height
- *triangulation*: planar subdivision whose bounded faces are triangles with vertices from A



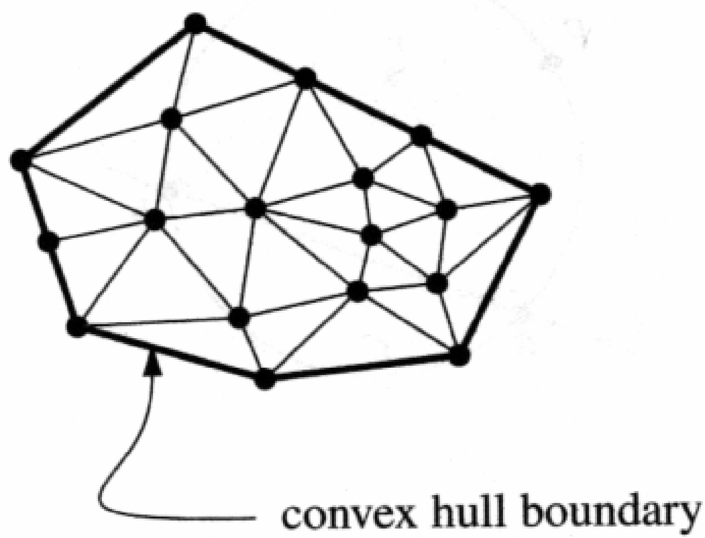


Triangulation: Formal Definition

- *maximal planar subdivision*: a subdivision S such that no edge connecting two vertices can be added to S without destroying its planarity
- *triangulation* of set of points P: a maximal planar subdivision whose vertices are elements of P

Triangulation is made of triangles

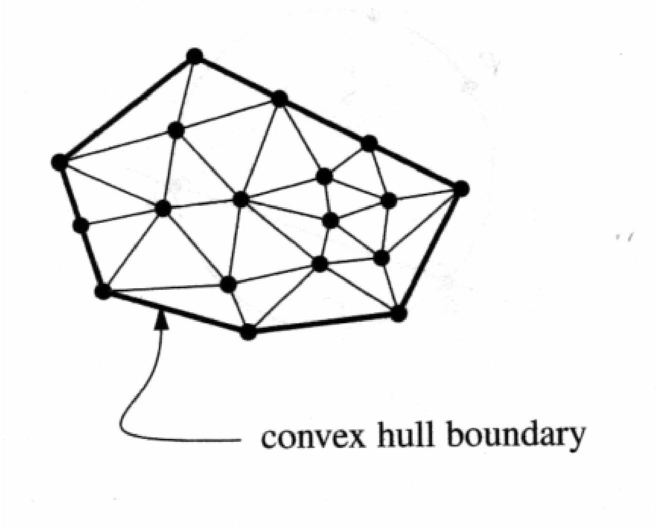
- Outer polygon must be convex hull
- Internal faces must be triangles, otherwise they could be triangulated further



Triangulation Details

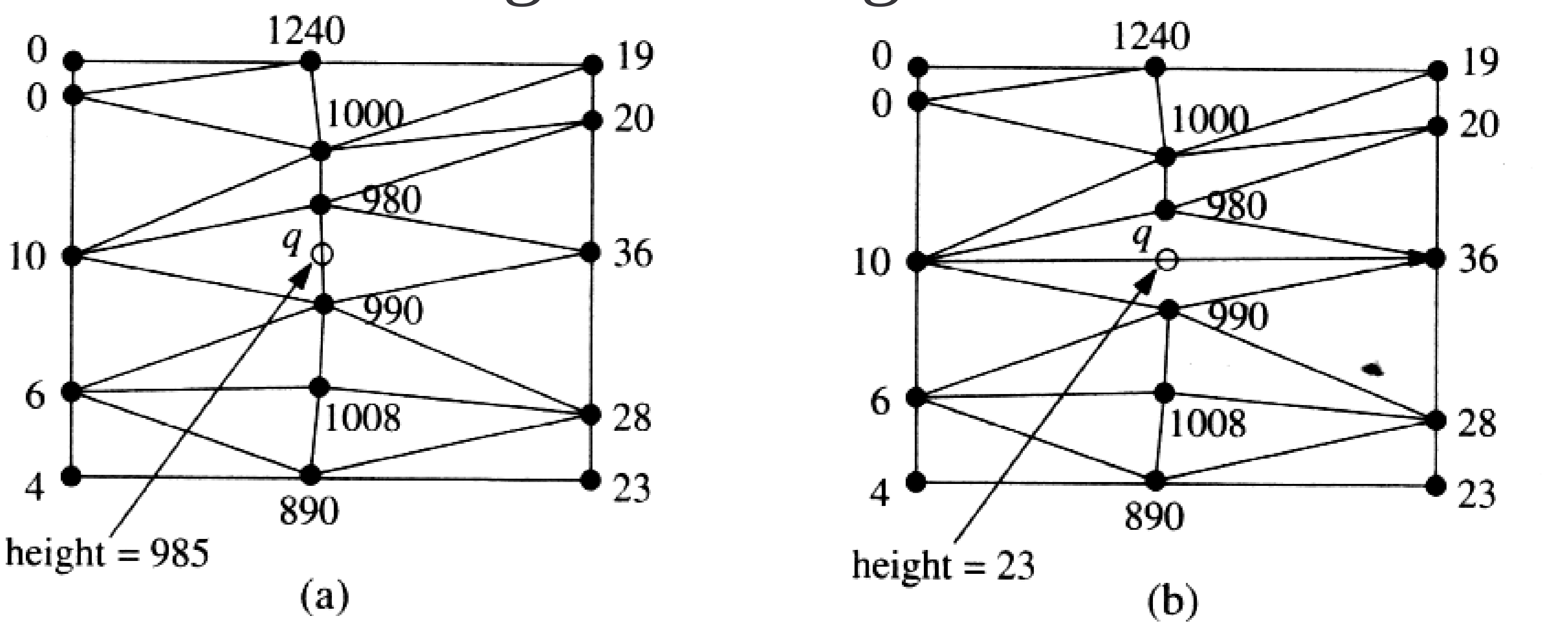
For P consisting of n points, all triangulations contain 2n-2-k triangles, 3n-3-k edges

- n = number of points in P
- k = number of points on convex hull of P



Terrain Problem, Revisited

- Some triangulations are “better” than others
- Avoid skinny triangles, i.e. maximize minimum angle of triangulation



(a) (b)

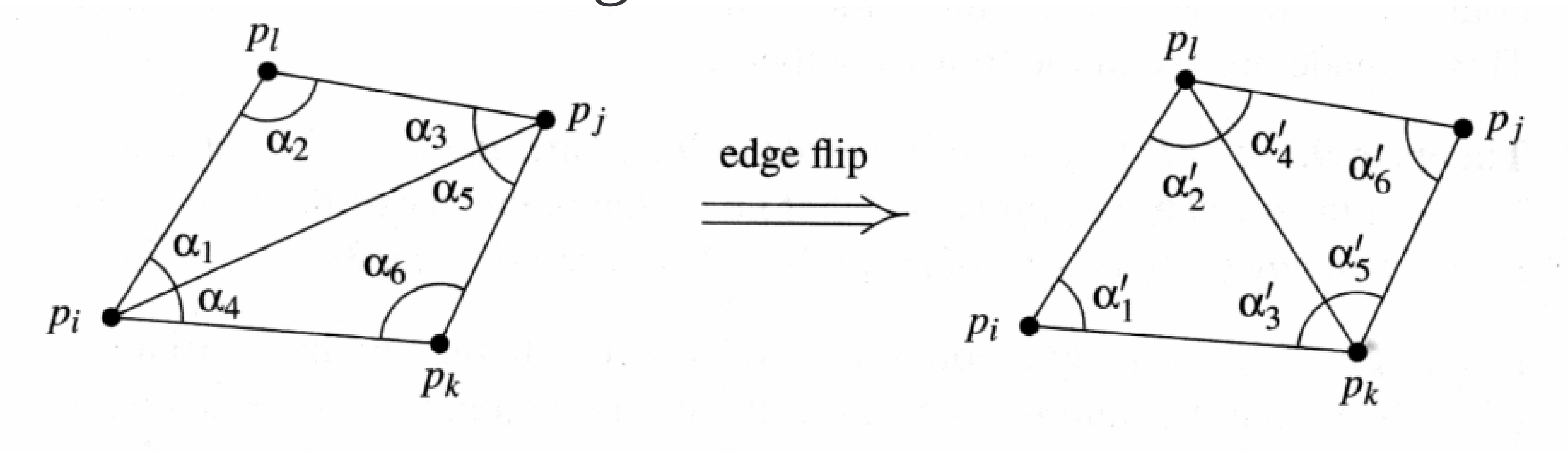
Angle Optimal Triangulations

- Create *angle vector* of the sorted angles of triangulation T, $(\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{3m}) = A(T)$ with α_1 being the smallest angle
- $A(T)$ is larger than $A(T')$ iff there exists an i such that $\alpha_j = \alpha'_j$ for all $j < i$ and $\alpha_i > \alpha'_i$
- Best triangulation is triangulation that is *angle optimal*, i.e. has the largest angle vector.

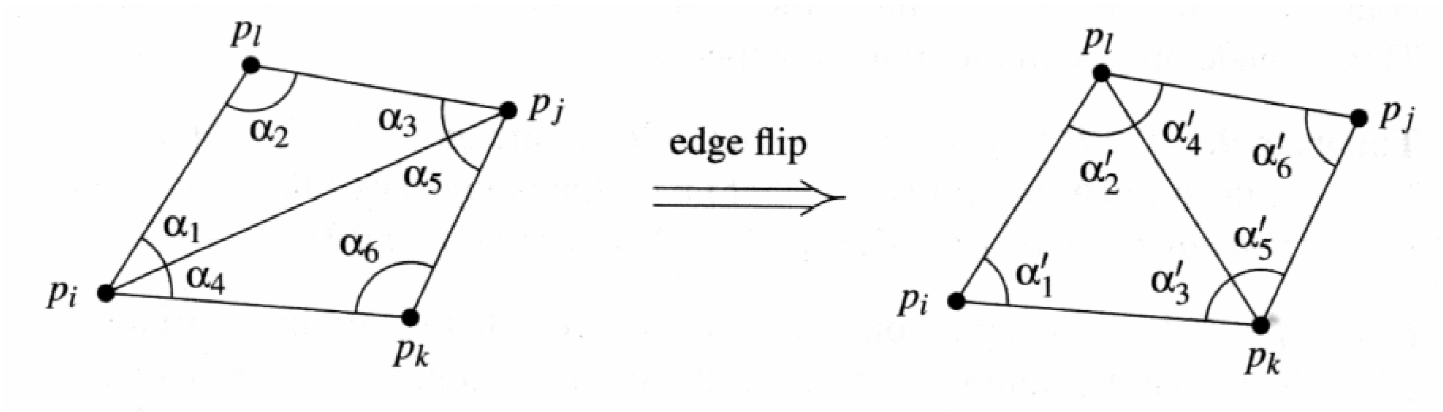
Angle Optimal Triangulations

Consider two adjacent triangles of T:

- If the two triangles form a convex quadrilateral, we could have an alternative triangulation by performing an *edge flip* on their shared edge.



Illegal Edges



- Edge *e* is illegal if:

$$\min_{1 \leqslant i \leqslant 6} \alpha_i < \min_{1 \leqslant i \leqslant 6} \alpha'_i.$$

- Only difference between *T* containing *e* and *T'* with *e* flipped are the six angles of the quadrilateral.

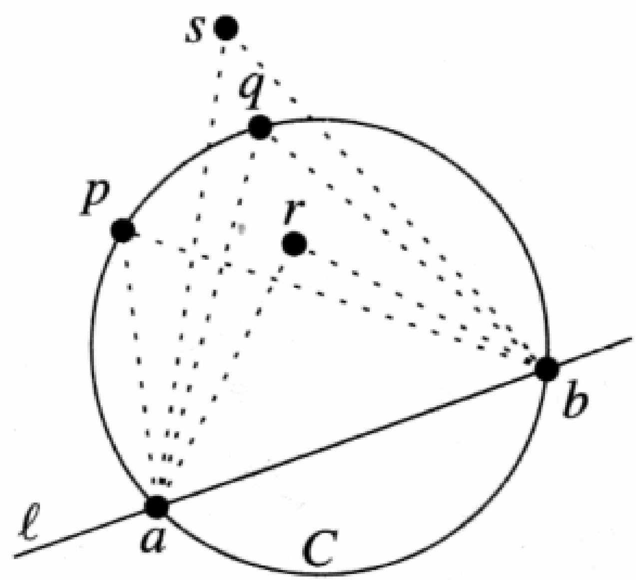
Illegal Triangulations

- If triangulation T contains an illegal edge e, we can make A(T) larger by flipping e.
- In this case, T is an *illegal triangulation*.

Thales's Theorem

- We can use *Thales's Theorem* to test if an edge is legal without calculating angles

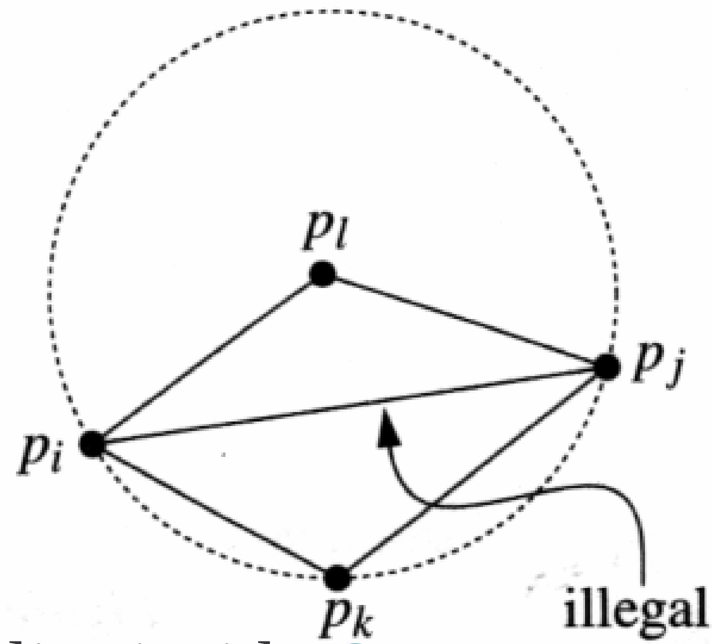
Let C be a circle, l a line intersecting C in points a and b and p, q, r, and s points lying on the same side of l. Suppose that p and q lie on C, that r lies inside C, and that s lies outside C. Then:



$$\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb.$$

Testing for Illegal Edges

- If p_i, p_j, p_k, p_l form a convex quadrilateral and do not lie on a common circle, exactly one of p_ip_j and p_kp_l is an illegal edge.



- The edge p_ip_j is illegal iff p_l lies inside C.
  - Proved using Thales's Theorem. E.g., the angle p_i-p_j-p_k is smaller than the angle p_i-p_l-p_k

Computing Legal Triangulations

1. Compute a triangulation of input points *P*.
2. Flip illegal edges of this triangulation until all edges are legal.

- Algorithm terminates because there is a finite number of triangulations.
- *Too slow to be interesting…*

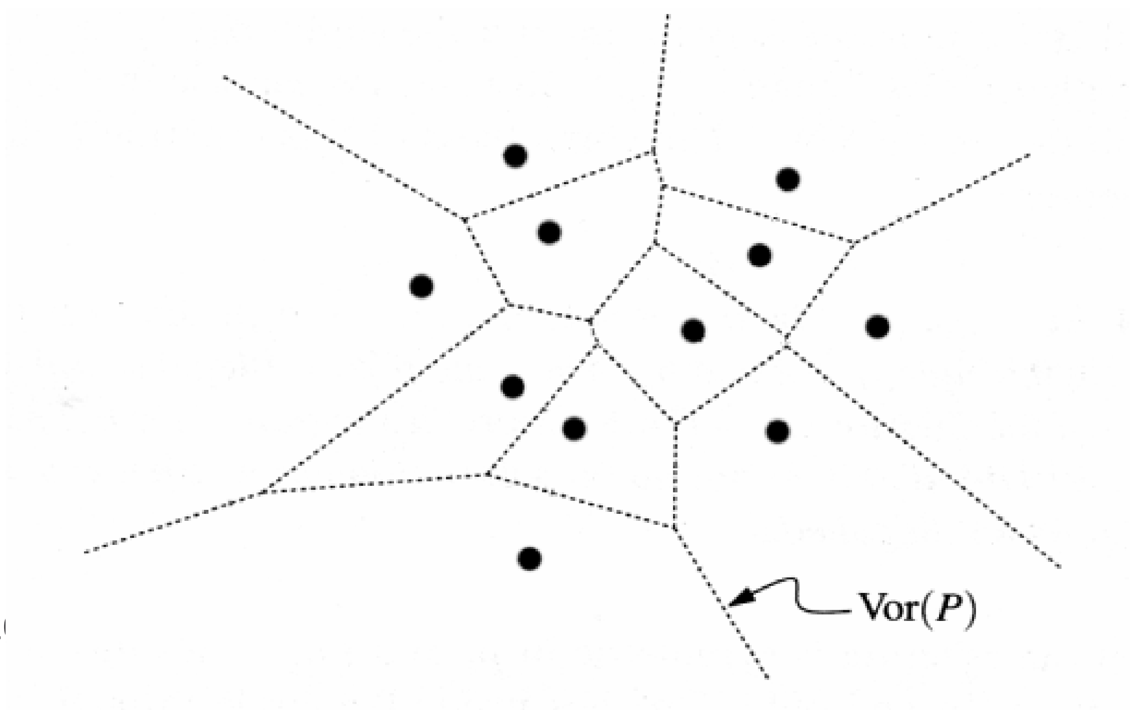
Sidetrack: Delaunay Graphs

- Before we can understand an interesting solution to the terrain problem, we need to understand Delaunay Graphs.
- Delaunay Graph of a set of points P is the dual graph of the Voronoi diagram of P

Delaunay Graphs

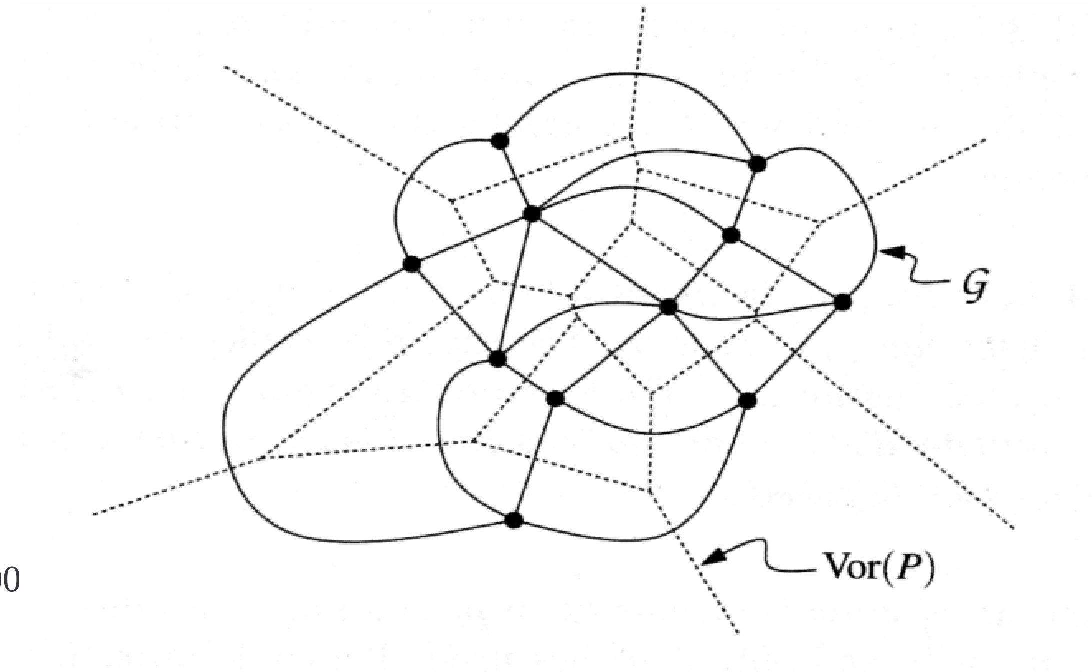
To obtain $\mathcal{DG}(P)$:

- Calculate $\mathcal{Vor}(P)$
- Place one vertex in each site of the $\mathcal{Vor}(P)$



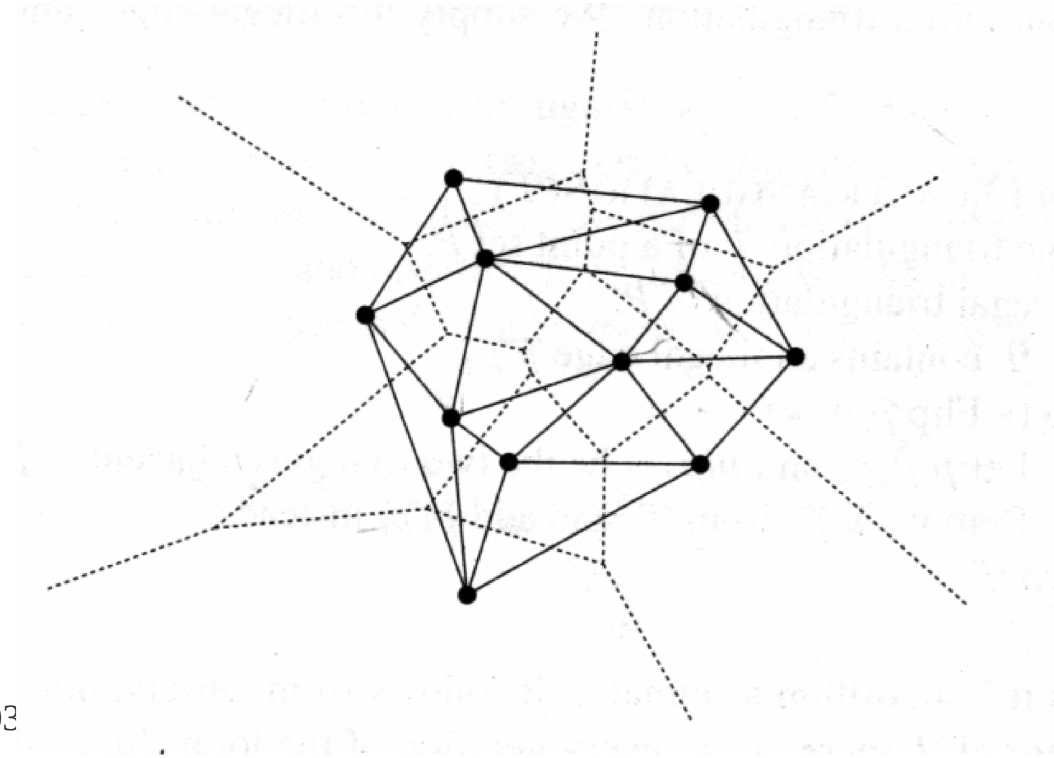
Constructing Delaunay Graphs

If two sites s_i and s_j share an edge (i.e., are adjacent), create an arc between v_i and v_j, the vertices located in sites s_i and s_j



Constructing Delaunay Graphs

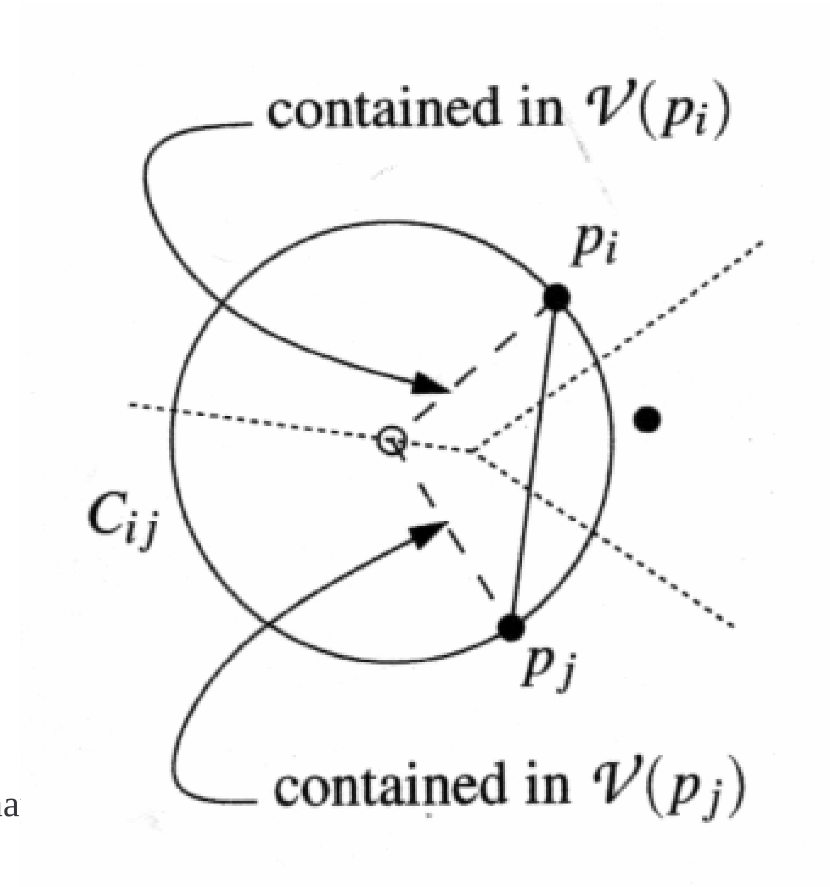
Finally, straighten the arcs into line segments. The resultant graph is $\mathcal{DG}$(P).



Properties of Delaunay Graphs

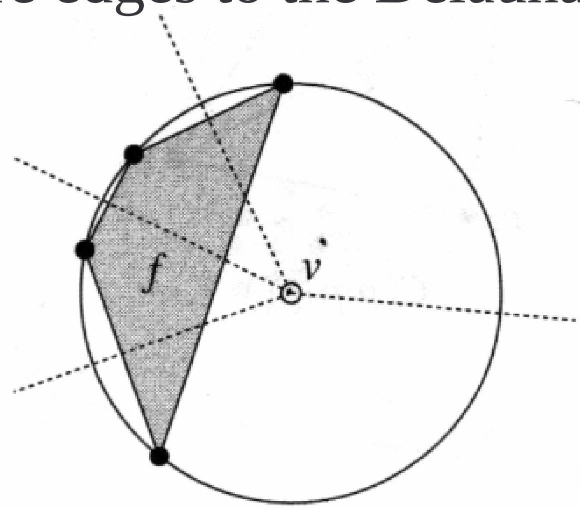
No two edges cross; $\mathcal{DG}(P)$ is a plane graph.

- Proved using the empty circle property



Delaunay Triangulations

- Some sets of more than 3 points of Delaunay graph may lie on the same circle.
- These points form empty convex polygons, which can be triangulated.
- *Delaunay Triangulation* is a triangulation obtained by adding 0 or more edges to the Delaunay Graph.

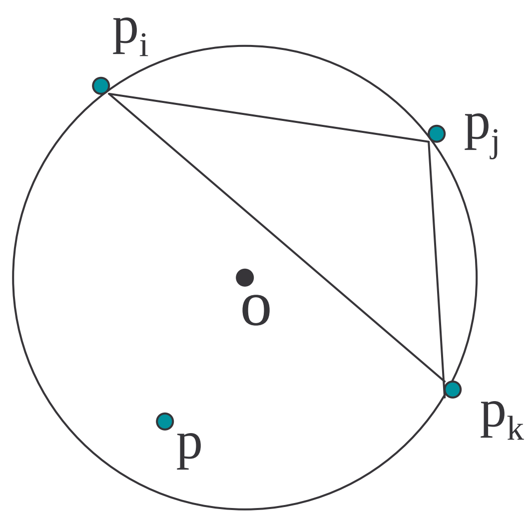


Properties of Delaunay Triangles

From the properties of Voronoi Diagrams…

- Three points $p_i, p_j, p_k \in P$ are vertices of the same face of the $\mathcal{DG}(P)$ iff the circle through p_i, p_j, p_k contains no point of P on its interior.

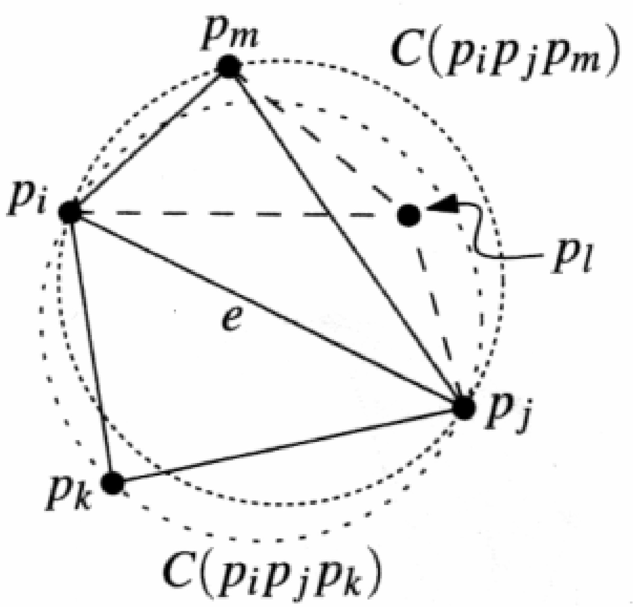
Proof:
- Assume there are other points inside the circle.
- Choose one point p inside the circle, and remove all other points but p_i, p_j, p_k. Note that, after the removal of points, p_i, p_j, p_k remains a triangle.
- Assume lies opposite p_j .
- p is closer to the center than are p_i, p_j, p_k. So, the center belongs to the interior of the Voronoi face of p.
- Consider a segment o-p_j. There is a point q on that segment that is equidistant to p_j and p, but its distance to p_i p_k is larger.
- Therefore, the Voronoi cells of p_j and p share an edge, so there is a Delaunay edge between p_j and p.
- But the Delaunay edges cannot intersect. QED.



Legal Triangulations, revisited

A triangulation $\mathcal{T}$ of P is legal iff $\mathcal{T}$ is a $\mathcal{DT}$(P).

- DT $\rightarrow$ Legal: Empty circle property and Thale's Theorem implies that all $\mathcal{DT}$ are legal
- Legal $\rightarrow$ DT: Proved on p. 190 from the definitions and via contradiction.



DT and Angle Optimal

The angle optimal triangulation is a $\mathcal{DT}$. Why?

- If P is in general position, $\mathcal{DT}(P)$ is unique and thus, is angle optimal.

What if multiple $\mathcal{DT}$ exist for P?

- Not all $\mathcal{DT}$ are angle optimal.
- By Thale's Theorem, the minimum angle of each of the $\mathcal{DT}$ is the same.
- Thus, all the $\mathcal{DT}$ are equally "good" for the terrain problem. All $\mathcal{DT}$ maximize the minimum angle.

Terrain Problem, revisited

Therefore, the problem of finding a triangulation that maximizes the minimum angle is reduced to the problem of finding a Delaunay Triangulation.

So how do we find the Delaunay Triangulation?

How do we compute $\mathcal{DT}(P)$?

- Compute $\mathcal{Vor}(P)$ then dualize into $\mathcal{DT}(P)$.
- We could also compute $\mathcal{DT}(P)$ using a randomized incremental method.

The rest is for the "curious"

Algorithm Overview

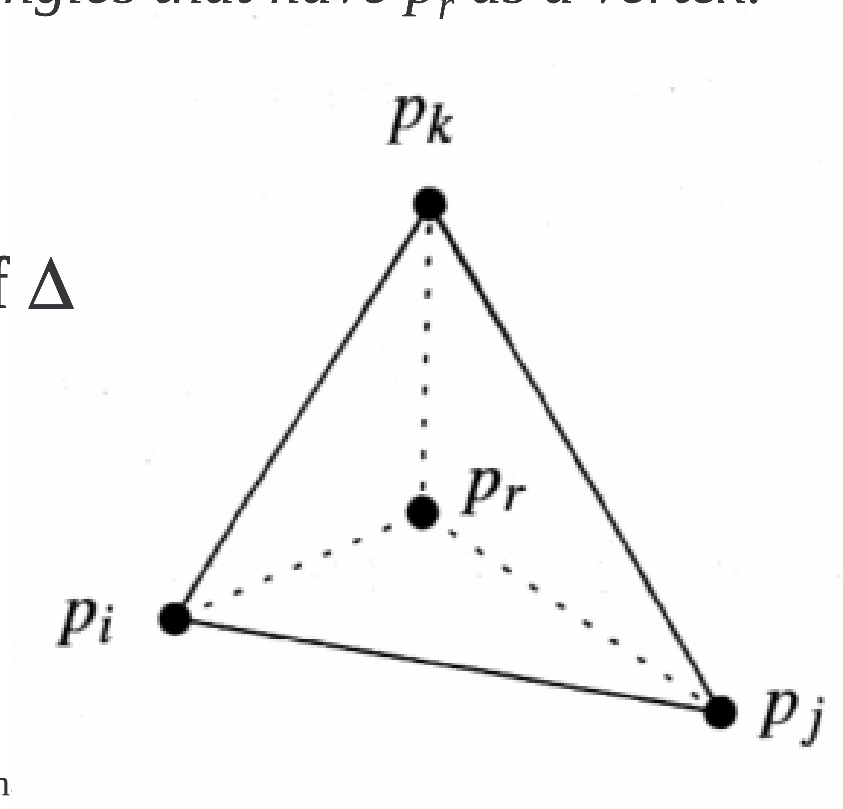
1. Initialize triangulation *T* with a “big enough” helper bounding triangle that contains all points *P*.
2. Randomly choose a point p_r from *P*.
3. Find the triangle Δ that p_r lies in.
4. Subdivide Δ into smaller triangles that have p_r as a vertex.
5. Flip edges until all edges are legal.
6. Repeat steps 2-5 until all points have been added to *T*.

Triangle Subdivision: Case 1 of 2

Assuming we have already found the triangle that p_r lives in, subdivide Δ into smaller triangles that have p_r as a vertex.

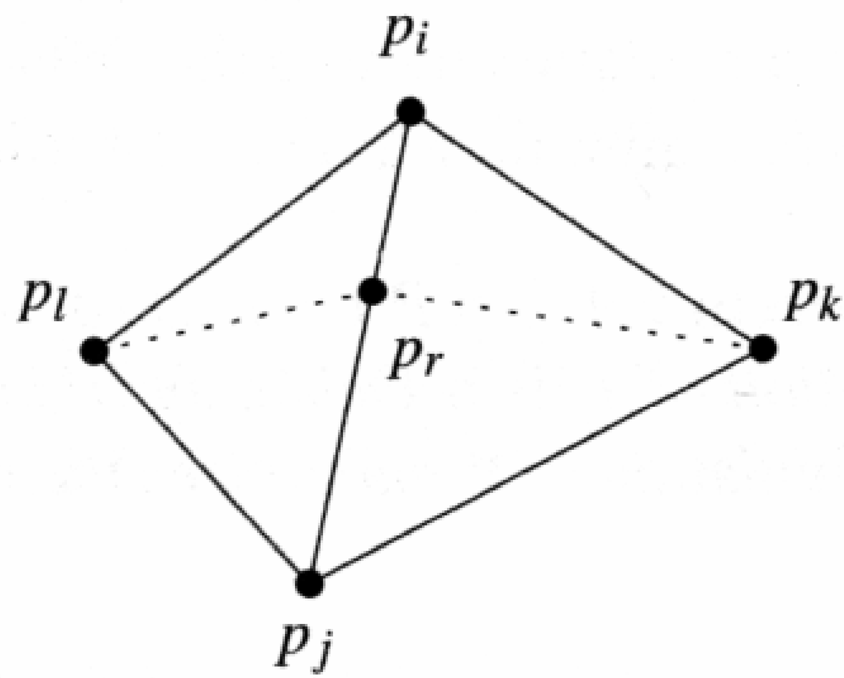
Two possible cases:

1) p_r lies in the interior of Δ



Triangle Subdivision: Case 2 of 2

2) p_r falls on an edge between two adjacent triangles

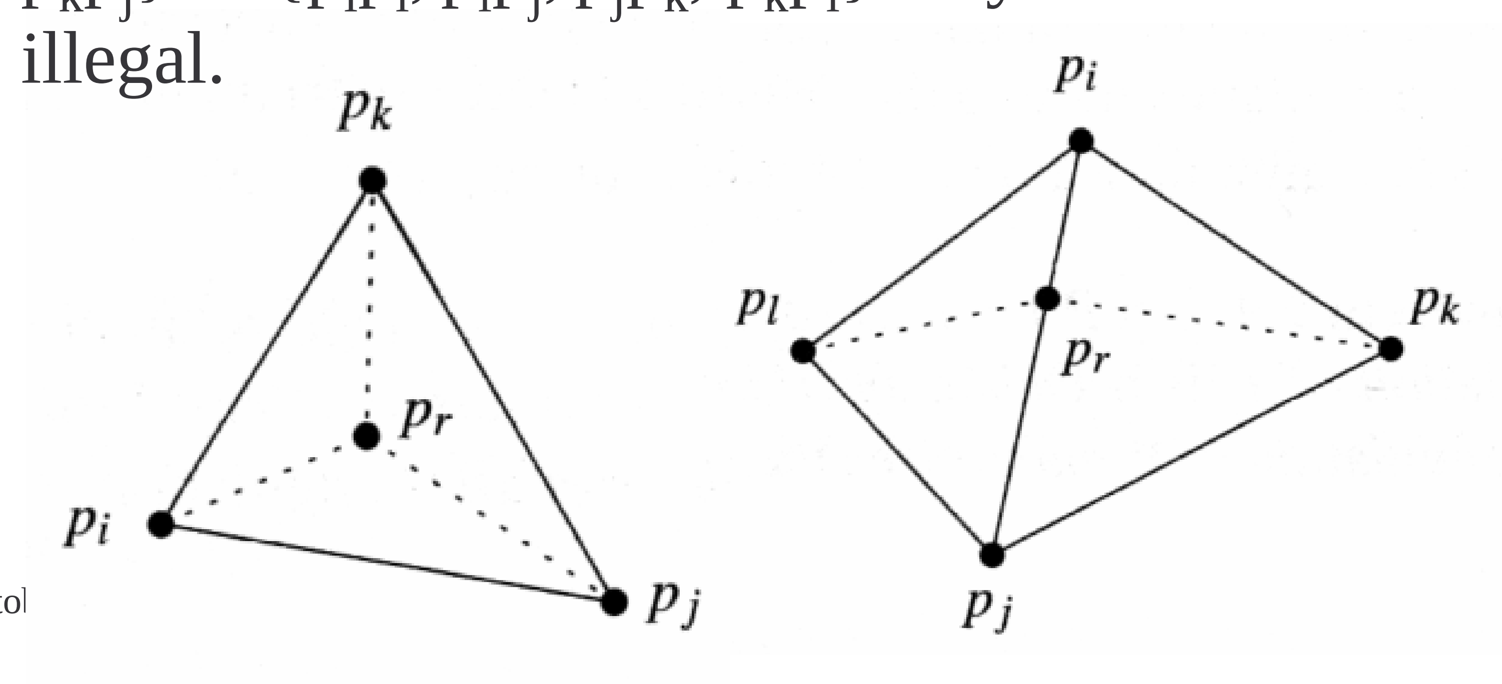


Which edges are illegal?

- Before we subdivided, all of our edges were legal.
- After we add our new edges, some of the edges of T may now be illegal, but which ones?

Outer Edges May Be Illegal

- An edge can become illegal only if one of its incident triangles changed.
- Outer edges of the incident triangles $\{p_j p_k, p_i p_k, p_k p_j\}$ or $\{p_i p_l, p_l p_j, p_j p_k, p_k p_i\}$ may have become illegal.



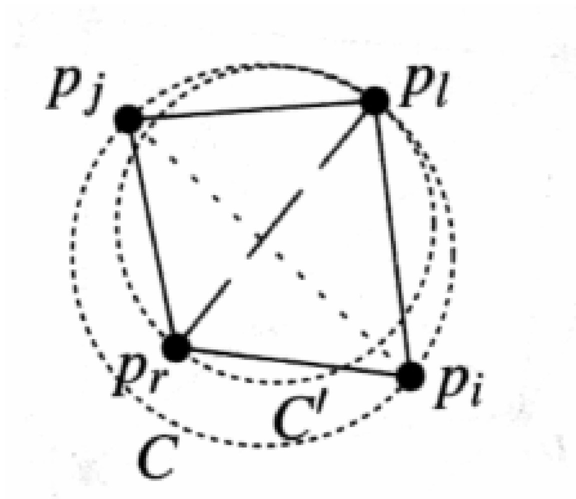
New Edges are Legal

Are the new edges (edges involving p_r) legal?

Consider **any** new edge $p_r p_l$.

Before adding $p_r p_l$,

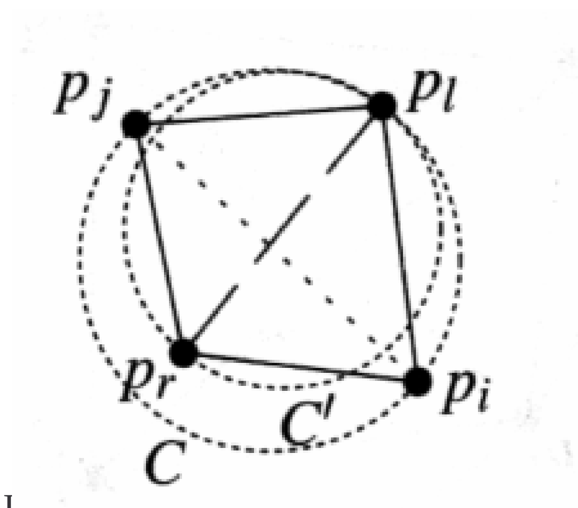
- p_l was part of some triangle $p_i p_j p_l$
- Circumcircle *C* of p_i, p_j, and p_l did not contain any other points of *P* in its interior



New edges incident to p_r are Legal

- If we shrink *C*, we can find a circle *C*' that passes through $p_r p_l$
- C' contains no points in its interior.
- Therefore, $p_r p_l$ is legal.

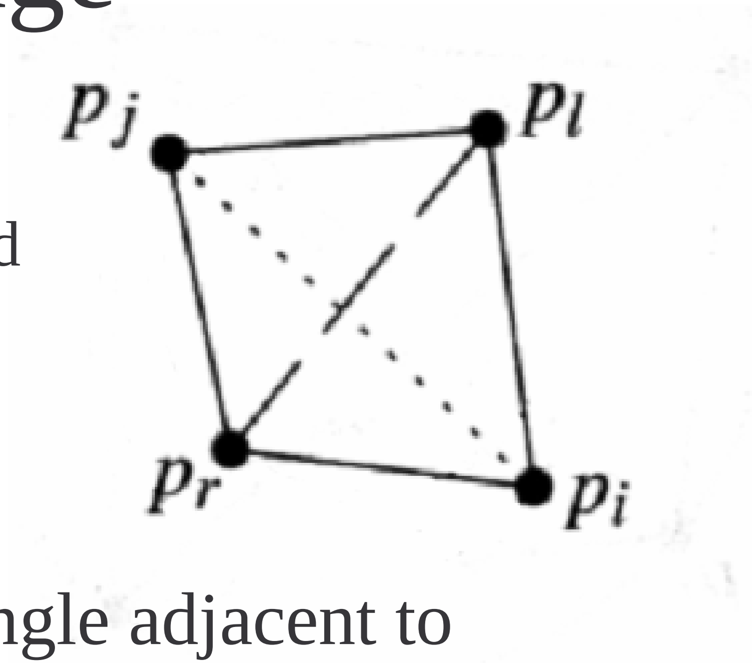
Any new edge incident p_r is legal.



Flip Illegal Edges

- Now that we know which edges have become illegal, we flip them.
- However, after the edges have been flipped, the edges incident to the new triangles may now be illegal.
- So we need to recursively flip edges…

LegalizeEdge



p_r = point being inserted

p_ip_j = edge that may need to be flipped

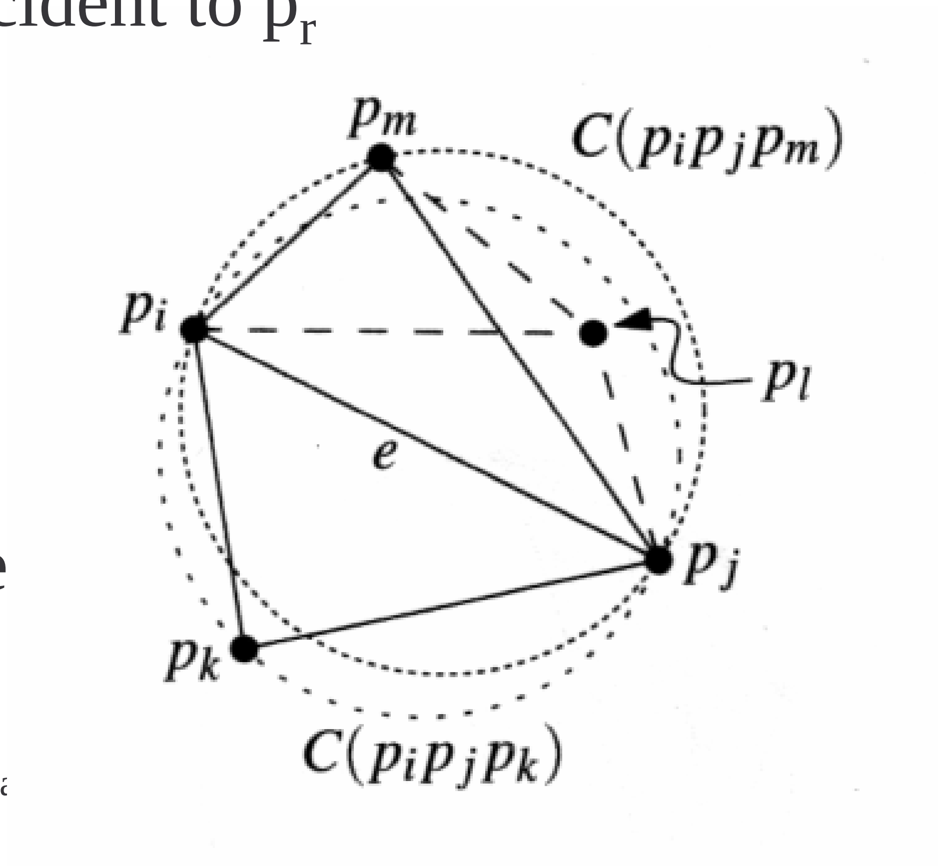
LEGALIZEEDGE(p_r, p_ip_j, T)

1. **if** p_ip_j is illegal
2. **then** Let $p_ip_jp_l$ be the triangle adjacent to $p_rp_ip_j$ along p_ip_j
3. Replace p_ip_j with p_rp_l
4. LEGALIZEEDGE(p_r, p_ip_l, T)
5. LEGALIZEEDGE(p_r, p_lp_j, T)

Flipped edges are incident to p_r

Notice that when LEGALIZEEDGE flips edges, these new edges are incident to p_r

- By the same logic as earlier, we can shrink the circumcircle of $p_i p_j p_l$ to find a circle that passes through p_r and p_l.
- Thus, the new edges are legal.



Bounding Triangle

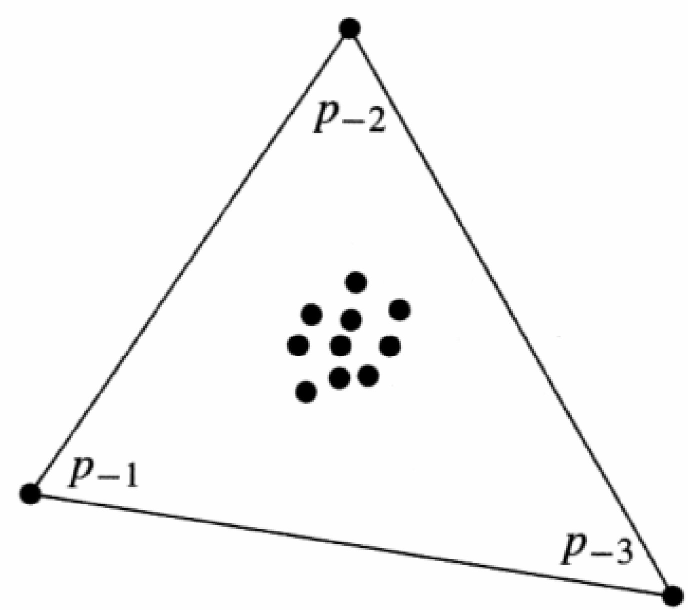
Remember, we skipped step 1 of our algorithm.

1. *Begin with a "big enough" helper bounding triangle that contains all points.*

Let $\{p_{-3}, p_{-2}, p_{-1}\}$ be the vertices of our bounding triangle.

"Big enough" means that the triangle:

- contains all points of P in its interior.
- will not destroy edges between points in P.



Considerations for Bounding Triangle

- We could choose large values for p_{-1}, p_{-2} and p_{-3}, but that would require potentially huge coordinates.
- Instead, we'll modify our test for illegal edges, to act as if we chose large values for bounding triangle.

Bounding Triangle

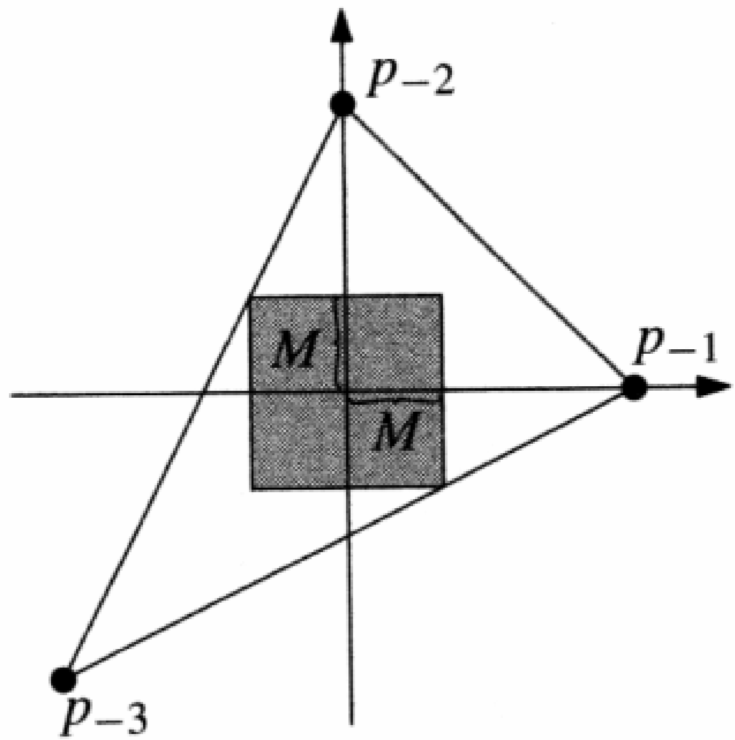
We'll *pretend* the vertices of the bounding triangle are at:

$p_{-1} = (3M, 0)$

$p_{-2} = (0, 3M)$

$p_{-3} = (-3M, -3M)$

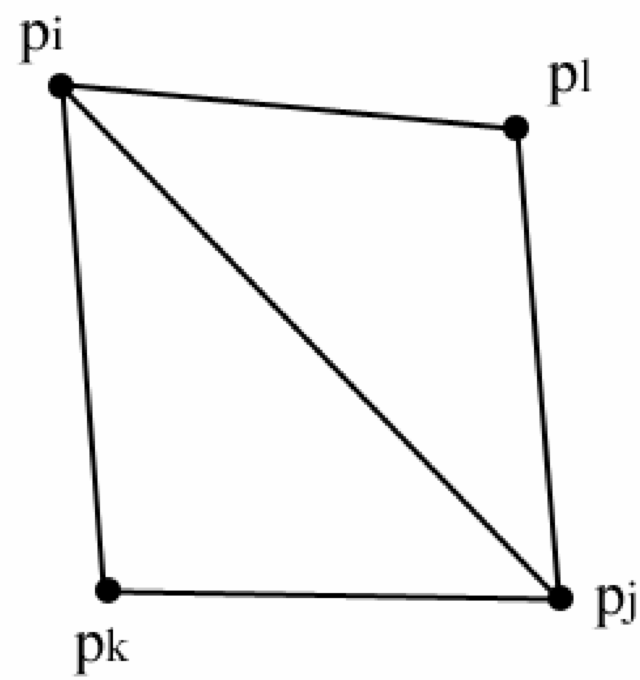
M = maximum absolute value of any coordinate of a point in P



Modified Illegal Edge Test

$p_i p_j$ is the edge being tested

p_k and p_l are the other two vertices of the triangles incident to $p_i p_j$



Our illegal edge test falls into one of 4 cases.

Illegal Edge Test, Case 1

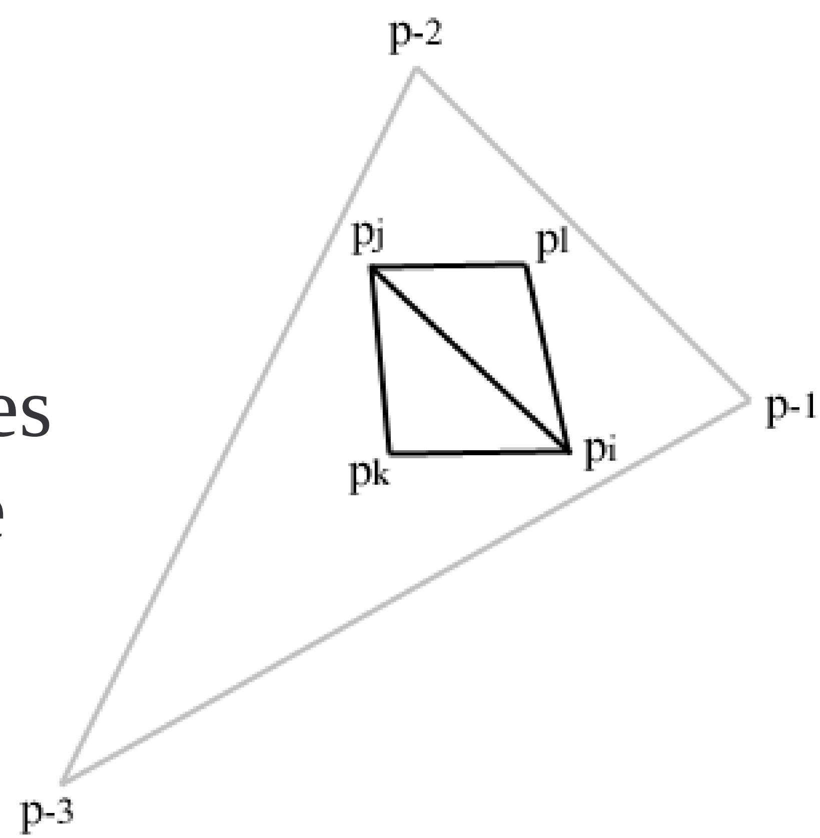
Case 1) Indices i and j are both negative

- $p_i p_j$ is an edge of the bounding triangle
- $p_i p_j$ is legal, want to preserve edges of bounding triangle

Illegal Edge Test, Case 2

Case 2) Indices i, j, k, and l are all positive.

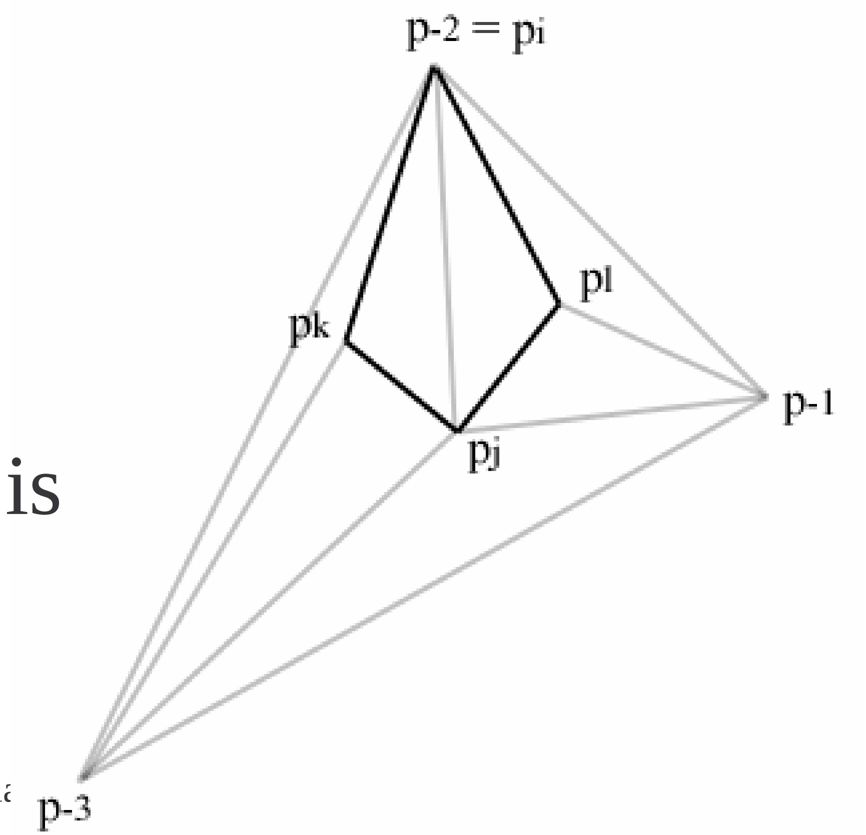
- This is the normal case.
- $p_i p_j$ is illegal iff p_l lies inside the circumcircle of $p_i p_j p_k$



Illegal Edge Test, Case 3

Case 3) Exactly one of i, j, k, l is negative

- We don't want our bounding triangle to destroy any Delaunay edges.
- If i or j is negative, $p_i p_j$ is illegal.
- Otherwise, $p_i p_j$ is legal.



Illegal Edge Test, Case 4

Case 4) Exactly two of i, j, k, l are negative.

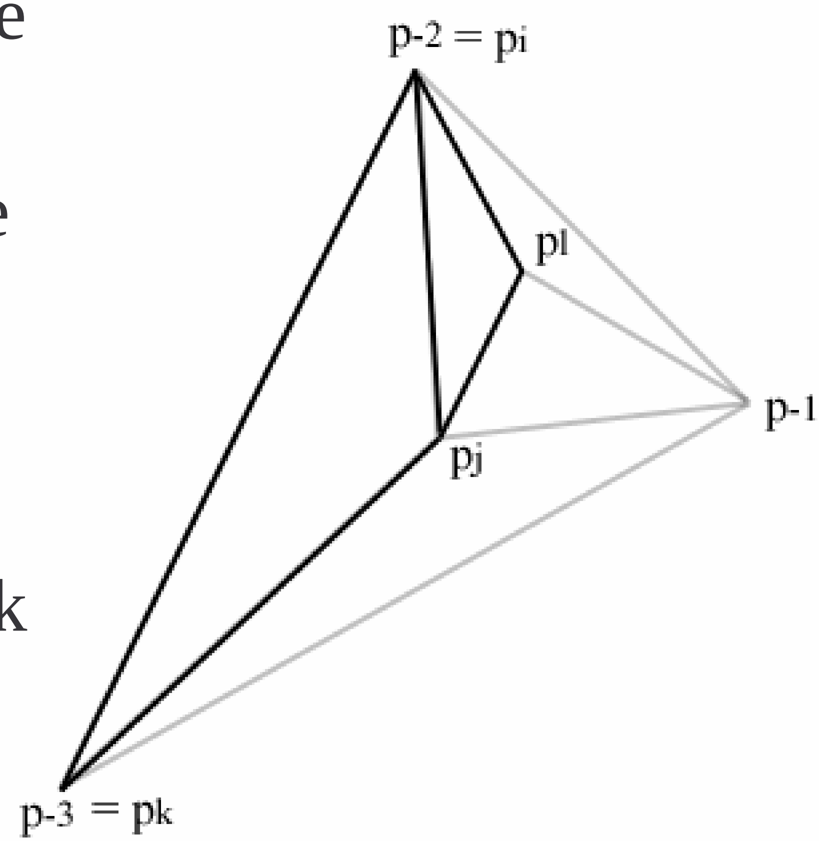
•k and l cannot both be negative (either p_k or p_l must be p_r)

•i and j cannot both be negative

•One of i or j and one of k or l must be negative

•If negative index of i and j is smaller than negative index of k and l, p_ip_j is legal.

•Otherwise p_ip_j is illegal.



Triangle Location Step

Remember, we skipped step 3 of our algorithm.

3. Find the triangle T that p_r lies in.

- Take an approach similar to Point Location approach.
- Maintain a point location structure $\mathcal{D}$, a directed acyclic graph.

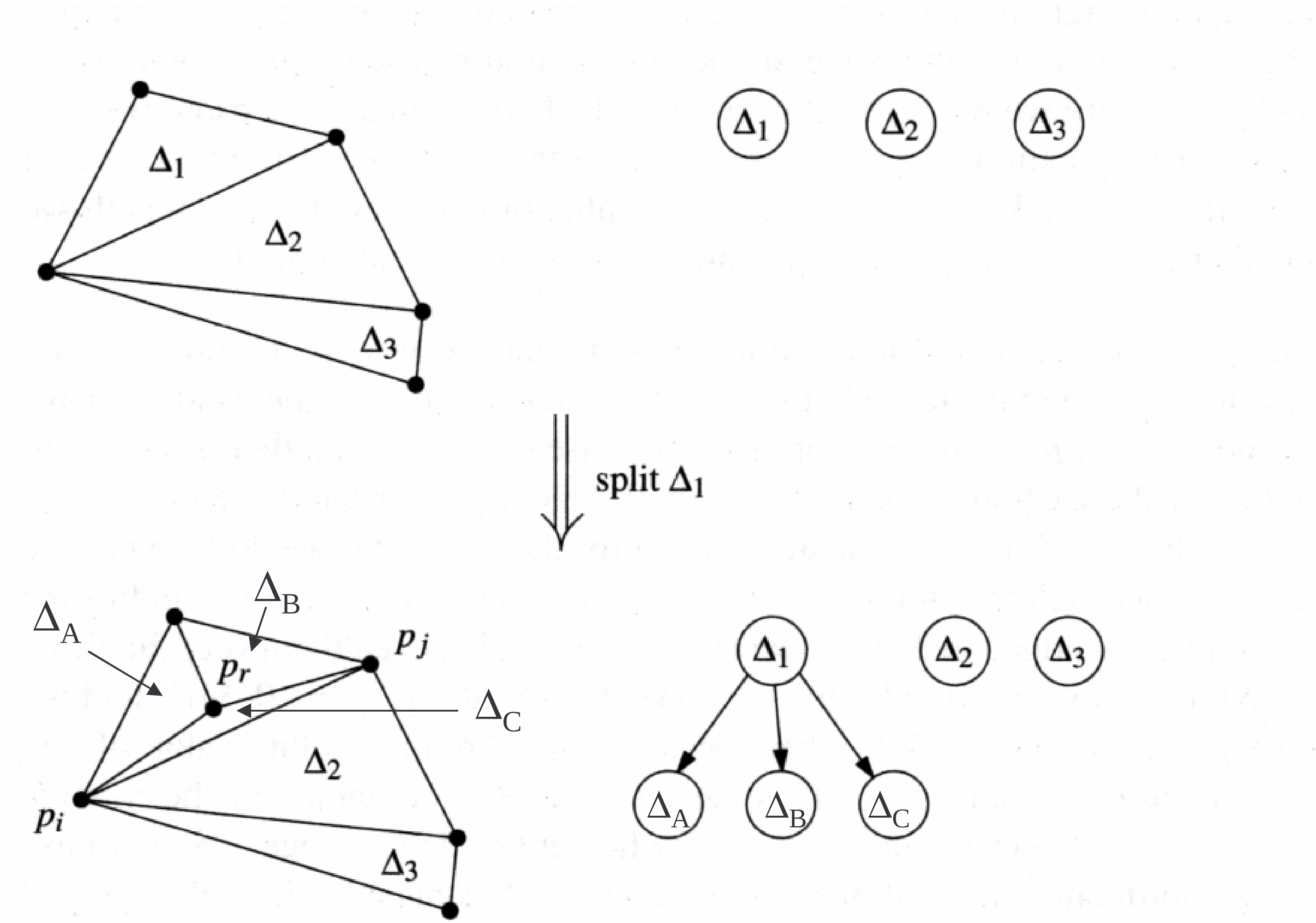
Structure of $\mathcal{D}$

- Leaves of $\mathcal{D}$ correspond to the triangles of the current triangulation.
- Maintain cross pointers between leaves of $\mathcal{D}$ and the triangulation.
- Begin with a single leaf, the bounding triangle $p_{-1}p_{-2}p_{-3}$

Subdivision and $\mathcal{D}$

- Whenever we split a triangle Δ_1 into smaller triangles Δ_a and Δ_b (and possibly Δ_c), add the smaller triangles to D as leaves of Δ_1

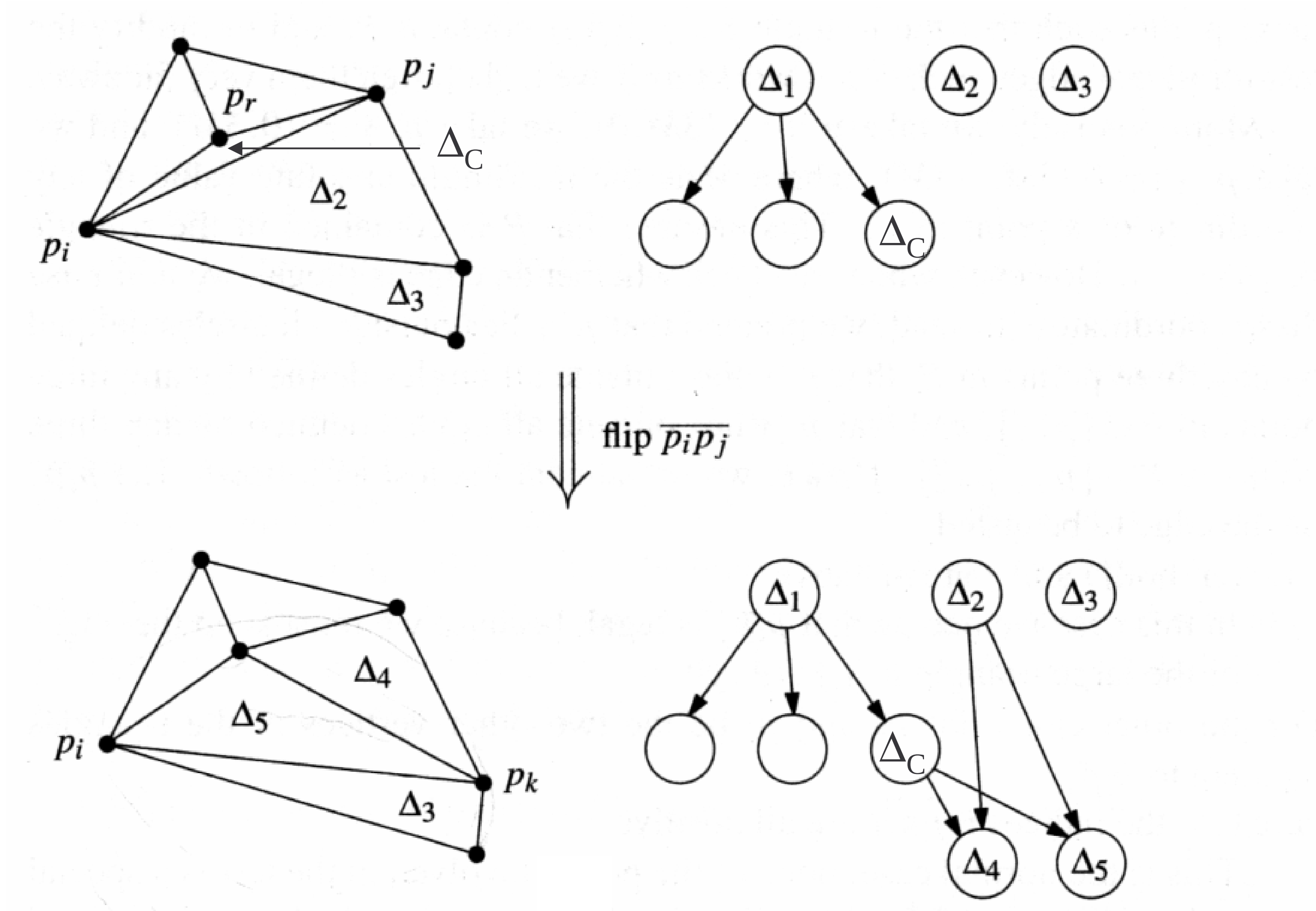
Subdivision and $\mathcal{D}$



Edge Flips and $\mathcal{D}$

- Whenever we perform an edge flip, create leaves for the two new triangles.
- Attach the new triangles as leaves of the two triangles replaced during the edge flip.

Edge Flips and $\mathcal{D}$



Searching $\mathcal{D}$

p_r = point we are searching with

1. Let the current node be the root node of $\mathcal{D}$.
2. Look at child nodes of current node. Check which triangle p_r lies in.
3. Let current node = child node that contains p_r
4. Repeat steps 2 and 3 until we reach a leaf node.

Searching $\mathcal{D}$

- Each node has at most 3 children.
- Each node in path represents a triangle in $\mathcal{D}$ that contains p_r
- Therefore, takes O(number of triangles in $\mathcal{D}$ that contain p_r)

Properties of $\mathcal{D}$

Notice that the:

- Leaves of $\mathcal{D}$ correspond to the triangles of the current triangulation.
- Internal nodes correspond to *destroyed triangles*, triangles that were in an earlier stage of the triangulation but are not present in the current triangulation.

Algorithm Overview

1. Initialize triangulation T with helper bounding triangle. Initialize $\mathcal{D}$.
2. Randomly choose a point p_r from P.
3. Find the triangle Δ that p_r lies in using $\mathcal{D}$.
4. Subdivide Δ into smaller triangles that have p_r as a vertex. Update $\mathcal{D}$ accordingly.
5. Call LEGALIZEEDGE on all possibly illegal edges, using the modified test for illegal edges. Update $\mathcal{D}$ accordingly.
6. Repeat steps 2-5 until all points have been added to T.

Analysis Goals

- Expected running time of algorithm is:

  $O(n \log n)$

- Expected storage required is:

  $O(n)$

First, some notation…

- $P_r = \{p_1, p_2, \ldots, p_r\}$
  – Points added by iteration r
- $\Omega = \{p_{-3}, p_{-2}, p_{-1}\}$
  – Vertices of bounding triangle
- $\mathcal{DG}_r = \mathcal{DG}(\Omega \cup P_r)$
  – Delaunay graph as of iteration r

Sidetrack: Expected Number of Δs

It will be useful later to know the expected number of triangles created by our algorithm…

Lemma 9.11 *Expected number of triangles created by* DELAUNAYTRIANGULATION *is* $9n+1$.

- In initialization, we create 1 triangle (bounding triangle).

Expected Number of Triangles

In iteration r where we add p_r,

- in the subdivision step, we create at most 4 new triangles. Each new triangle creates one new edge incident to p_r
- each edge flipped in LEGALIZEEDGE creates two new triangles and one new edge incident to p_r

Expected Number of Triangles

Let k = number of edges incident to p_r after insertion of p_r, the degree of p_r

- We have created at most 2(k-3)+3 triangles.
- -3 and +3 are to account for the triangles created in the subdivision step

The problem is now to find the expected degree of p_r

Expected Degree of p_r

Use backward analysis:

- Fix P_r, let p_r be a random element of P_r
- $\mathcal{DG}_r$ has 3(r+3)-6 edges
- Total degree of $P_r \leq 2[3(r+3)-9] = 6r$

E[degree of random element of P_r] ≤ 6

Triangles created at step r

Using the expected degree of pr, we can find the expected number of triangles created in step r.

$\deg(p_r, \mathcal{DG}_r)$ = degree of p_r in $\mathcal{DG}_r$

$$
\begin{aligned}
\mathrm{E}\left[\text{number of triangles created in step } r\right] &\leqslant \mathrm{E}\left[2\deg(p_r, \mathcal{DG}_r) - 3\right] \\
&= 2\mathrm{E}\left[\deg(p_r, \mathcal{DG}_r)\right] - 3 \\
&\leqslant 2 \cdot 6 - 3 = 9
\end{aligned}
$$

Expected Number of Triangles

Now we can bound the number of triangles:

$\leq$ 1 initial Δ + Δs created at step 1 + Δs created at step 2 + … + Δs created at step n

$\leq 1 + 9n$

Expected number of triangles created is $9n+1$.

Storage Requirement

- $\mathcal{D}$ has one node per triangle created
- 9n+1 triangles created
- O(n) expected storage

Expected Running Time

Let's examine each step…

1. *Begin with a "big enough" helper bounding triangle that contains all points.*

   O(1) time, executed once = O(1)

2. *Randomly choose a point p_r from P.*

   O(1) time, executed n times = O(n)

3. *Find the triangle Δ that p_r lies in.*

   Skip step 3 for now…

Expected Running Time

4. *Subdivide Δ into smaller triangles that have p_r as a vertex.*

   O(1) time executed n times = O(n)

5. *Flip edges until all edges are legal.*

   In total, expected to execute a total number of times proportional to number of triangles created = O(n)

Thus, total running time without point location step is O(n).

Point Location Step

- Time to locate point p_r is

  O(number of nodes of $\mathcal{D}$ we visit)

  + O(1) for current triangle
- Number of nodes *of* $\mathcal{D}$ we visit

  = number of destroyed triangles that contain p_r
- A triangle is destroyed by p_r if its circumcircle contains p_r

We can charge each triangle visit to a Delaunay triangle whose circumcircle contains p_r

Point Location Step

K(Δ) = subset of points in *P* that lie in the circumcircle of Δ

- When $p_r \in K(\Delta)$, charge to Δ.
- Since we are iterating through *P*, each point in K(Δ) can be charged at most once.

Total time for point location:

$$O(n + \sum_{\Delta} \mathrm{card}(K(\Delta))),$$

Point Location Step

We want to have O(n log n) time, therefore we want to show that:

$$\sum_{\Delta} \operatorname{card}(K(\Delta)) = O(n \log n),$$

Point Location Step

Introduce some notation…

- $\mathcal{T}_r$ = set of triangles of $\mathcal{DG}(\Omega \cup P_r)$
- $\mathcal{T}_r \setminus \mathcal{T}_{r-1}$ triangles created in stage r

Rewrite our sum as:

$$\sum_{r=1}^{n} \left(\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \operatorname{card}(K(\Delta)) \right).$$

Point Location Step

More notation…

$k(P_r, q)$ = number of triangles $\Delta \in \mathcal{T}_r$ such that q is contained in Δ

$k(P_r, q, p_r)$ = number of triangles $\Delta \in \mathcal{T}_r$ such that q is contained in Δ and p_r is incident to Δ

Rewrite our sum as:

$$\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \operatorname{card}(K(\Delta)) = \sum_{q \in P \setminus P_r} k(P_r, q, p_r).$$

Point Location Step

Find the $E[k(P_r, q, p_r)]$ then sum later…

- Fix P_r, so $k(P_r, q, p_r)$ depends only on p_r.
- Probability that p_r is incident to a triangle is $3/r$

Thus:

$$\mathrm{E}\left[k(P_r, q, p_r)\right] \leqslant \frac{3k(P_r, q)}{r}.$$

Point Location Step

Using:

$$\mathrm{E}\left[k(P_r, q, p_r)\right] \leqslant \frac{3k(P_r, q)}{r}.$$

We can rewrite our sum as:

$$\mathrm{E}\Big[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta))\Big] \leqslant \frac{3}{r} \sum_{q \in P \setminus P_r} k(P_r, q).$$

Point Location Step

Now find $E[k(P_r, p_{r+1})]$…

- Any of the remaining n-r points is equally likely to appear as p_{r+1}

So:

$$E\left[k(P_r, p_{r+1})\right] = \frac{1}{n-r} \sum_{q \in P \setminus P_r} k(P_r, q).$$

Point Location Step

Using:

$$\mathrm{E}\left[k(P_r, p_{r+1})\right] = \frac{1}{n-r} \sum_{q \in P \setminus P_r} k(P_r, q).$$

We can rewrite our sum as:

$$\mathrm{E}\Big[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta))\Big] \leqslant 3\left(\frac{n-r}{r}\right) \mathrm{E}\left[k(P_r, p_{r+1})\right].$$

Point Location Step

Find $k(P_r, p_{r+1})$

- number of triangles of $\mathcal{T}_r$ that contain p_{r+1}
- these are the triangles that will be destroyed when p_{r+1} is inserted; $\mathcal{T}_r \setminus \mathcal{T}_{r+1}$
- Rewrite our sum as:

$$\mathrm{E}\Big[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta))\Big] \leqslant 3\left(\frac{n-r}{r}\right)\mathrm{E}\big[\mathrm{card}(\mathcal{T}_r \setminus \mathcal{T}_{r+1})\big].$$

Point Location Step

Remember, number of triangles in triangulation of n points with k points on convex hull is 2n-2-k

- $\mathcal{T}_m$ has 2(m+3)-2-3=2m+1
- $\mathcal{T}_{m+1}$ has two more triangles than Tm

Thus, card($\mathcal{T}_\mathrm{r} \setminus \mathcal{T}_{\mathrm{r}+1}$)

= card(triangles destroyed by p_r)

= card(triangles created by p_r) – 2

= card($\mathcal{T}_{\mathrm{r}+1} \setminus \mathcal{T}_\mathrm{r}$) - 2

We can rewrite our sum as:

$$\mathrm{E}\Big[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \operatorname{card}(K(\Delta))\Big] \leqslant 3\left(\frac{n-r}{r}\right)\Big(\mathrm{E}\big[\operatorname{card}(\mathcal{T}_{r+1} \setminus \mathcal{T}_r)\big] - 2\Big).$$

Point Location Step

Remember we fixed P_r earlier…

- Consider all P_r by averaging over both sides of the inequality, but the inequality comes out identical.

E[*number of triangles created by* p_r]

= E[*number of edges incident to* p_{r+1} *in* $\mathcal{T}_{r+1}$]

= 6

Therefore:

$$\mathrm{E}\Big[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta))\Big] \leqslant 12\left(\frac{n-r}{r}\right).$$

Analysis Complete

$$\mathrm{E}\Big[\sum_{\Delta \in \mathcal{T}_r \setminus \mathcal{T}_{r-1}} \mathrm{card}(K(\Delta))\Big] \leqslant 12\left(\frac{n-r}{r}\right).$$

If we sum this over all *r*, we have shown that:

$$\sum_{\Delta} \mathrm{card}(K(\Delta)) = O(n \log n),$$

And thus, the algorithm runs in O(*n* log *n*) time.