Point Location

(most slides by Sergi Elizalde and David Pritchard)

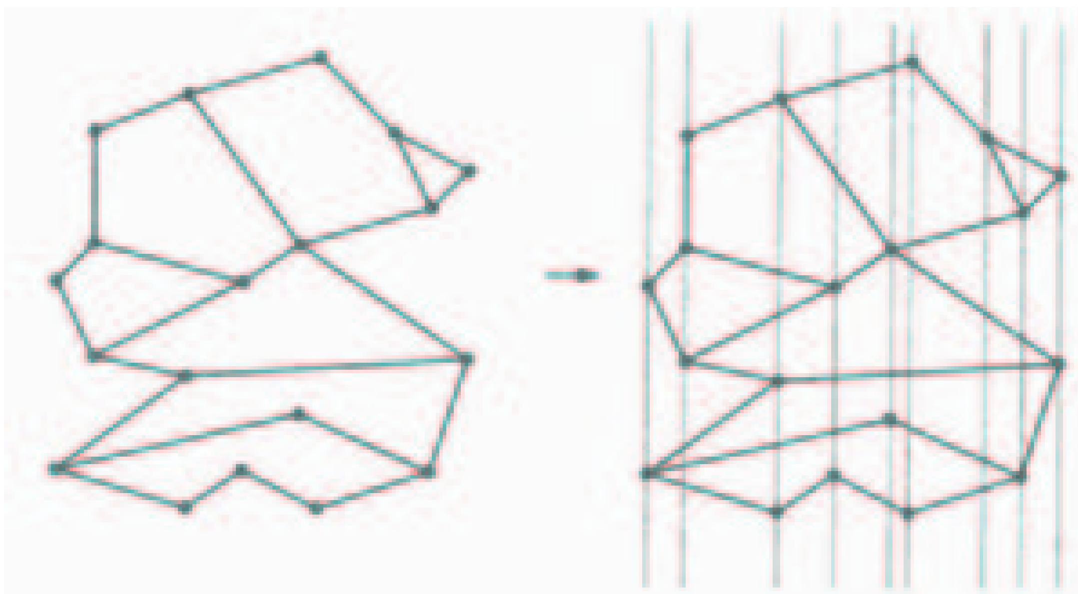
Point Location

Definition

- Given: a planar subdivision S
- Goal: build a data structure that, given a query point, determines which face of the planar subdivision that point lies in
- Details: planar subdivision given by:
  - Vertices, directed edges and faces
  - Perimeters of polygons stored in doubly linked lists
  - Can switch between faces, edges and vertices in constant time

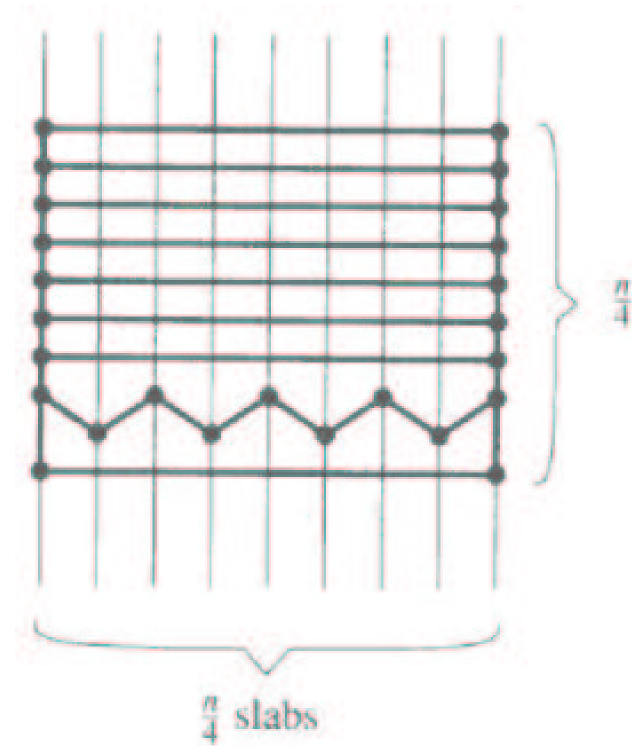
First attempt

- Want to divide the plane into easily manageable sections.
- Idea: Divide the graph into slabs, by drawing a vertical line through every vertex of the graph
- Given the query point, do binary search in the proper slab



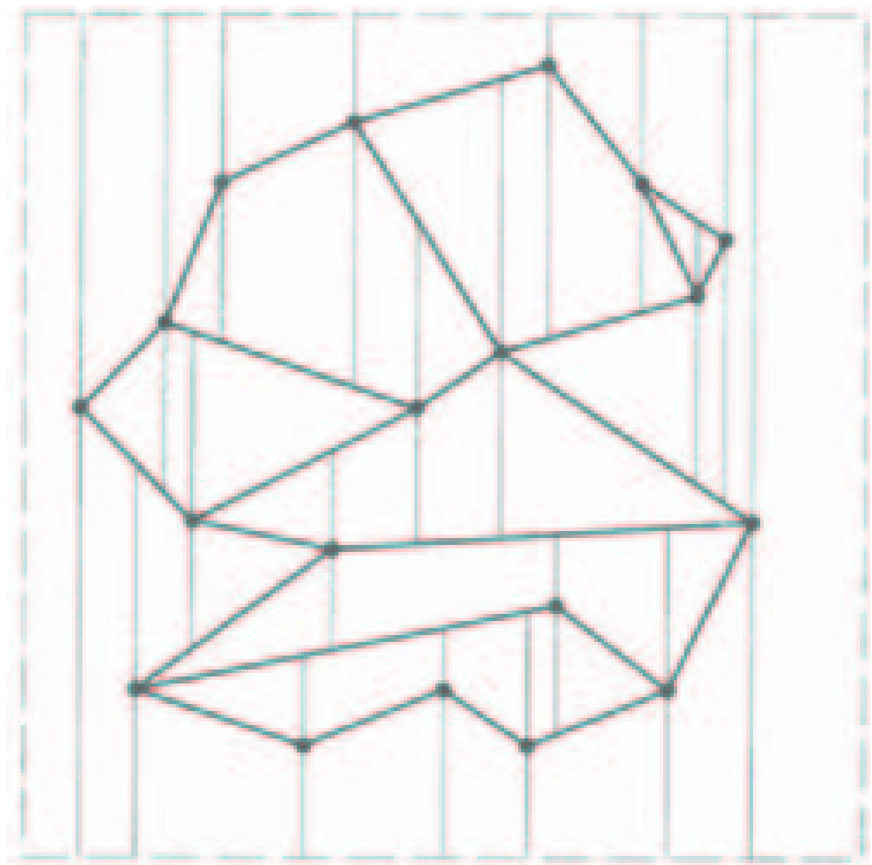
Analysis

- Query time: O(log n)
- Space: $O(n^2)$
- As a few people in the audience observed, the space can be reduced to O(n) by using "persistent" data structures. See 6.854, Lecture 5 for details.



Second attempt

- Too much splitting!
- Idea: stop the splitting lines at the first segment of the subdivision
- We get a *trapezoidal decomposition* T(S) of S
- The number of edges still O(n)



Assumptions/Simplifications

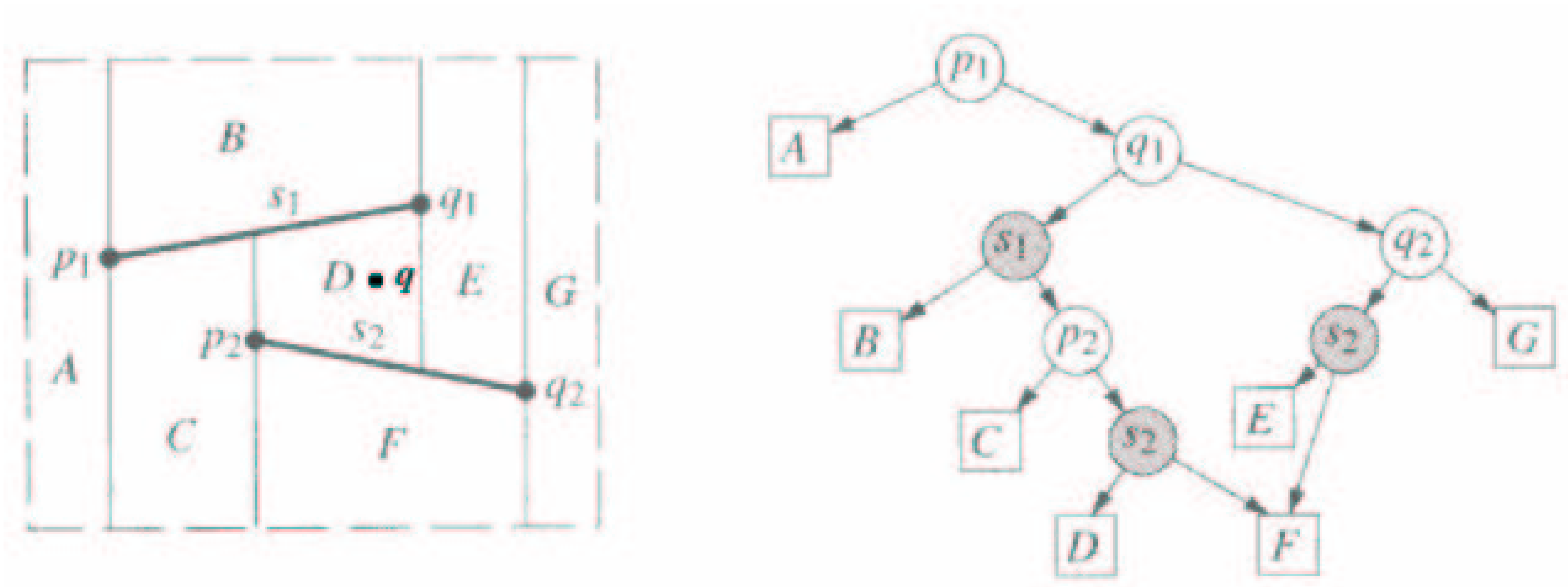
- Add a bounding box that contains S
- Assume that the x-coordinates of coordinates and query are distinct

1. Randomly rotate the plane, or
2. Use lexicographic order

Answering the query

- Build a decision tree:
  - Leaves: individual trapezoids
  - Internal nodes: YES/NO queries:
    - *point query*: does *q* lie to the left or the right of a given point?
    - *segment query*: does *q* lie above or below a given line segment?

Decision tree: Example

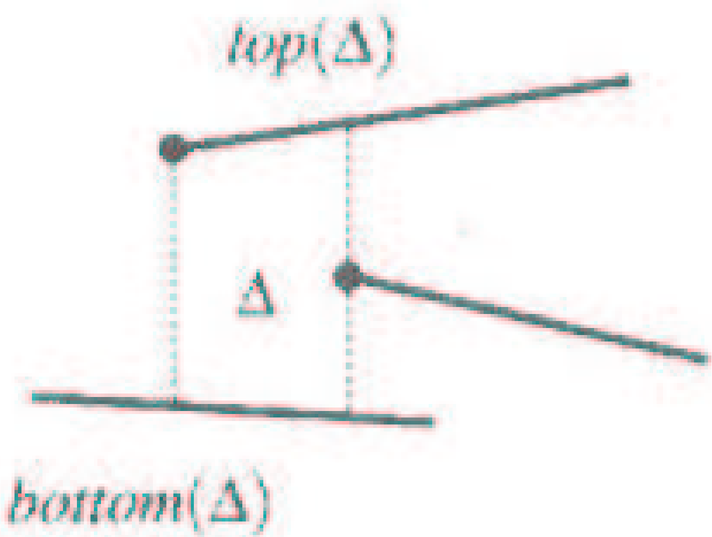


DT Construction: Overview

1. Initialization: create a T with the bounding box R as the only trapezoid, and corresponding DT D
2. Compute a random permutation of segments $s_1 \ldots s_n$
3. For each segment s_i:
   A. Find the set of trapezoids in T properly intersected by s_i
   B. Remove them from T and replace them by the new trapezoids that appear because of the insertion of s_i
   C. Remove the leaves of D for the old trapezoids and create leaves for the new ones + update links

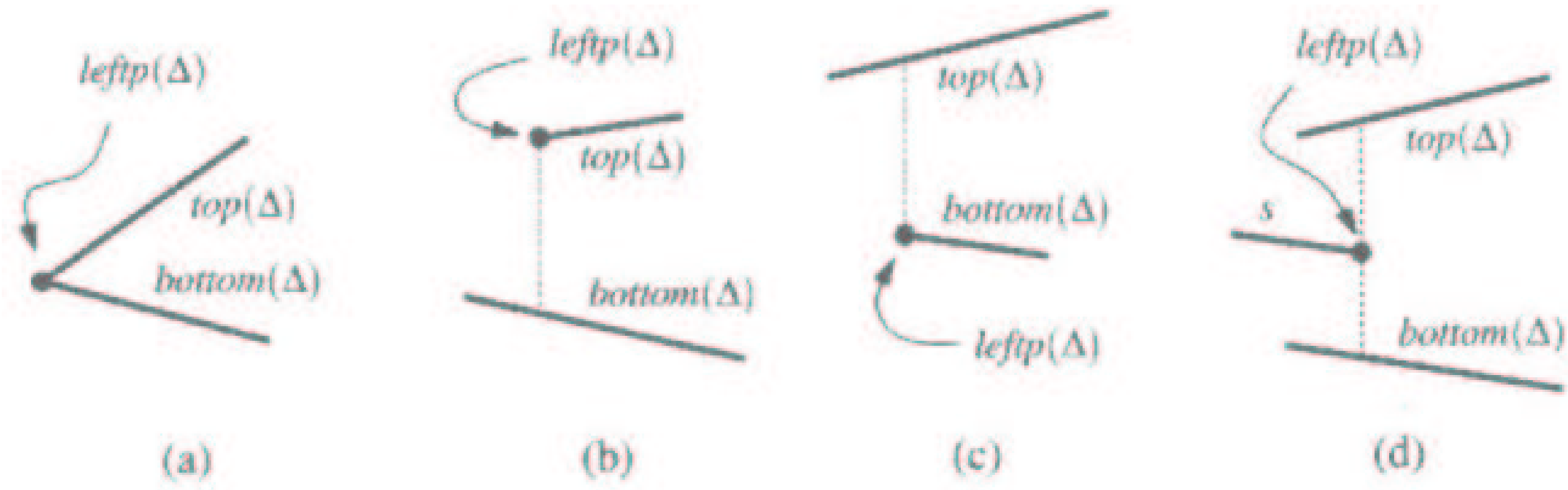
Some notation

Segments top(Δ) and bottom(Δ) :



Some notation, ctd.

Points leftp(Δ) and rightp(Δ):

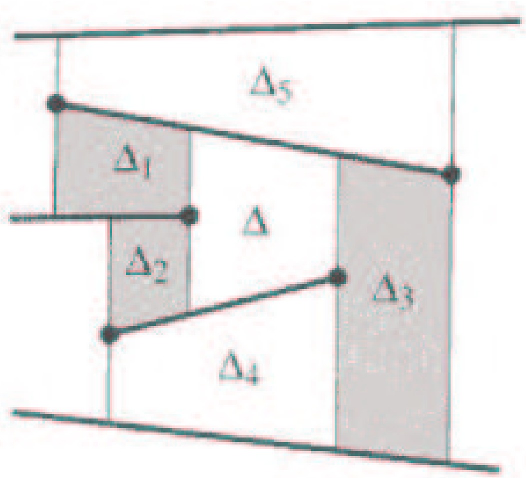


Each Δ is defined by top(Δ), bottom(Δ), leftp(Δ), rightp(Δ)

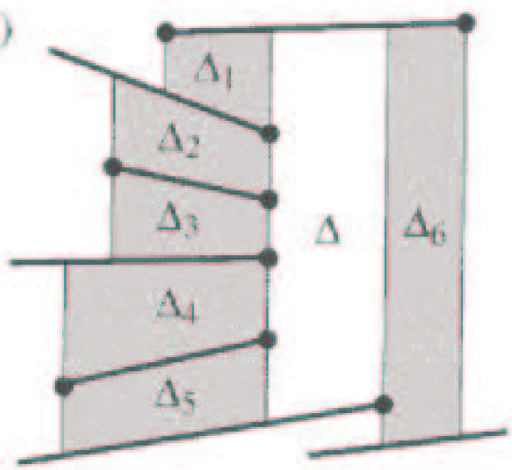
Some notation, ctd.

- Two trapezoids are *adjacent* if they share a vertical boundary
- How many trapezoids can be adjacent to Δ ?

(i)

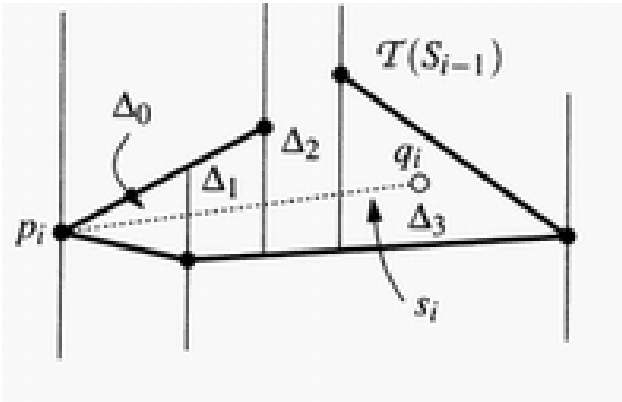


(ii)



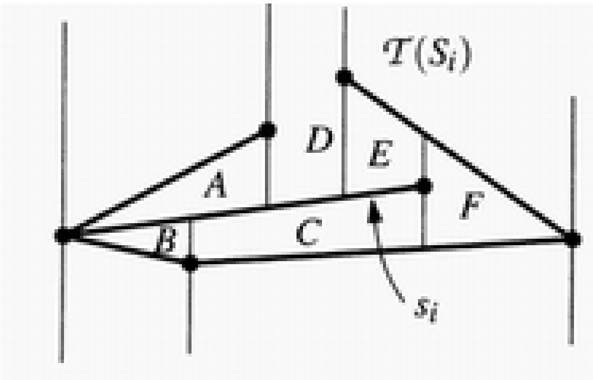
Adding new segment s_i

- Let $\Delta_0 \ldots \Delta_k$ be the trapezoids intersected by s_i (left to right)
- To find them:
  - Δ_0 is the trapezoid containing the left endpoint p of s_i – find it by querying the data structure built so far
  - Δ_{j+1} must be a right neighbor of Δ_j



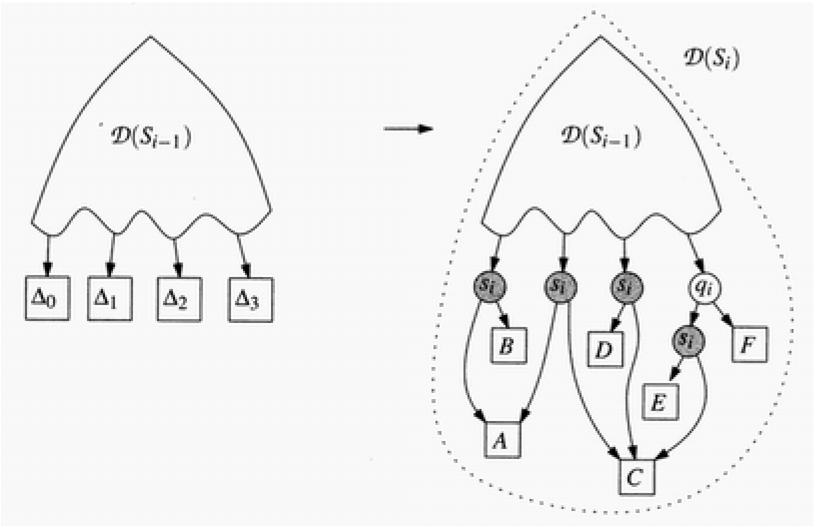
Updating T

- Draw vertical extensions through the endpoints of s_i that were not present, partitioning $\Delta_0 \ldots \Delta_k$
- Shorten the vertical extensions that now end at s_i, merging the appropriate trapezoids



Updating D

- Remove the leaves for $\Delta_0 \ldots \Delta_k$
- Create leaves for the new trapezoids
- If Δ_0 has the left endpoint p of s_i in its interior, replace the leaf for Δ_0 with a point node for p and a segment node for s_i (similarly with Δ_k)
- Replace the leaves of the other trapezoids with single segment nodes for s_i
- Make the outgoing edges of the inner nodes point to the correct leaves



Analysis

- Theorem: In the expectation we have
  - Running time: $O(n \log n)$
  - Storage: $O(n)$
  - Query time $O(\log n)$ for a fixed q

Expected Query Time

- Fix a query point q, and consider the path in D traversed by the query.
- Define
  - $S_i = \{s_1, s_2, ..., s_i\}$
  - X_i = number of nodes added to the search path for q during iteration i
  - P_i = probability that some node on the search path of q is created in iteration i
  - $\Delta_q(S_i)$ = trapezoid containing q in $T(S_i)$
- From our construction, $X_i \leq 3$; thus $E[X_i] \leq 3P_i$
- Note that $P_i = Pr[\Delta_q(S_i) <> \Delta_q(S_{i-1})]$

Expected Query Time ctd.

- What is $P_i = \Pr[\Delta_q(S_i) <> \Delta_q(S_{i-1})]$?
- Backward analysis: How many segments in S_i affect $\Delta_q(S_i)$ when they are removed?
- At most 4
- Since they have been chosen in random order, each one has probability $1/i$ of being s_i
- Thus $P_i \leq 4/i$
- $E[\sum_i X_i] = \sum_i E[X_i] \leq \sum_i 3P_i \leq \sum_i 12/i = O(\log n)$

Expected Storage

- Number of nodes bounded by $O(n) + \sum_i k_i$, where k_i= number of new trapezoids created in iteration i
- Define $d(\Delta,s)$ to be 1 iff Δ disappears from $T(S_i)$ when s removed from S_i
- We have $\sum_{s\in Si} \sum_{\Delta \in T(Si)} d(\Delta,s) \le 4|T(S_i)|=O(i)$
- $E[k_i]= [\sum_{s\in Si} \sum_{\Delta \in T(Si)} d(\Delta,s)]/i =O(1)$

Expected Time

- The time needed to insert s_i is $O(k_i)$ plus the time needed to locate the left endpoint of s_i in $T(S_i)$
- Expected running time = $O(n \log n)$

Extensions

- Can obtain worst-case $O(\log n)$ query time
  - Show $O(\log n)$ for a fixed query holds with probability $1-1/(Cn^2)$ for large C
  - There are $O(n^2)$ truly different queries