# Geometric Pattern Matching

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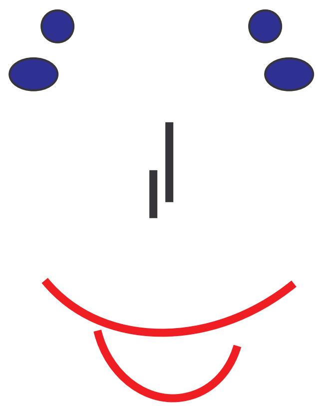
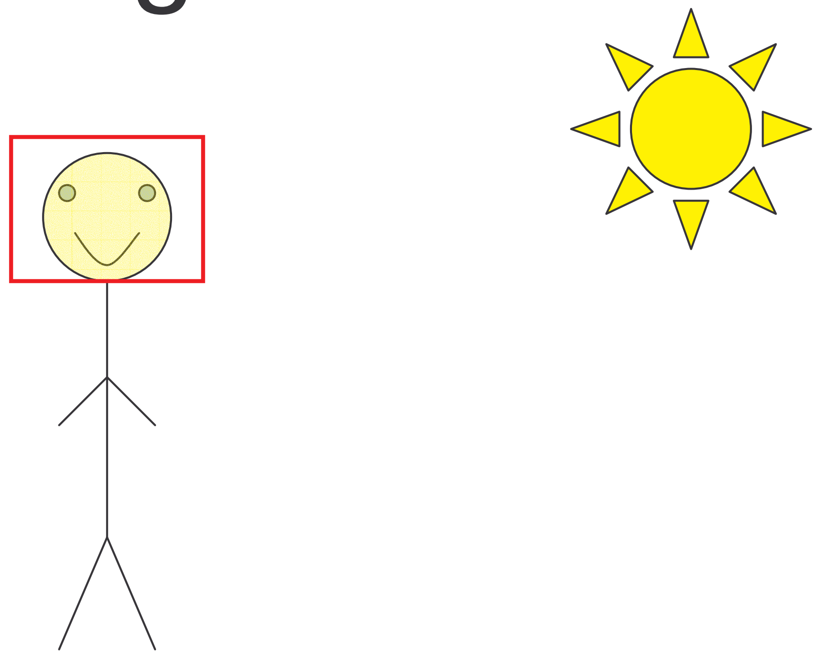
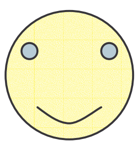
# Face Recognition



# Matching in a Scene





# Formalization: Shapes

- Today:
  - A shape is a set A of points in $R^2$
  - $|A|=n$
- In general, A could consist of segments etc.

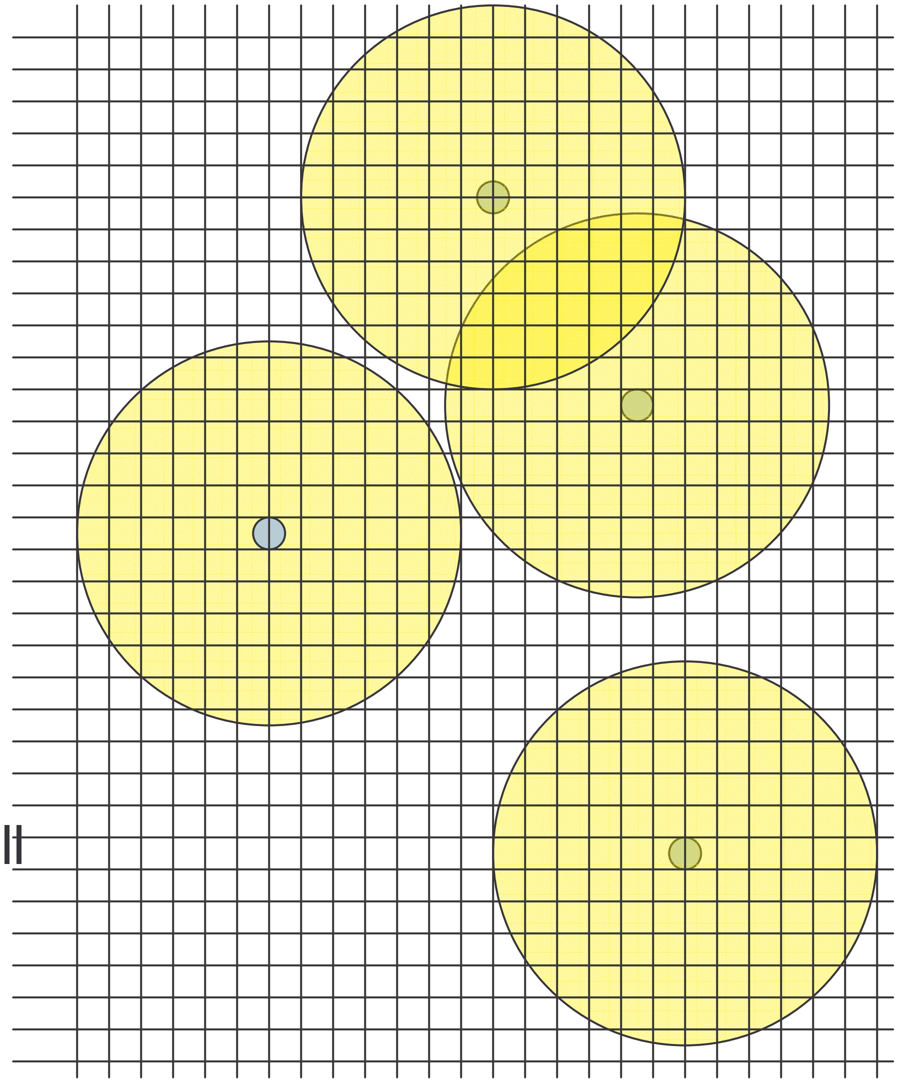
# Formalization: (Dis)similarity

- Hausdorff distance
  - $DH(A,B)=\max_{a\in A} \min_{b\in B} ||a-b||$
  - $H(A,B)=\max[\, DH(A,B), DH(B,A) \,]$
- Earth-Mover Distance
  - Minimum cost of a one-to-one matching between A and B
  - $EMD(A,B)=\min_{f:A\rightarrow B,} \sum_{a\in A} ||a-f(a)||$, f is 1:1
- Bottleneck matching
  - $BM(A,B)= \min_{f:A\rightarrow B,} \max_{a\in A} ||a-f(a)||$, f is 1:1

# Computing H(A,B)

- Given A,B, how fast can we compute H(A,B) ?
  - We will compute DH(A,B)
  - Construct a Voronoi diagram V for B
  - Construct a point location structure for V
  - For each a in A, find its NN in B
- Total time: O(n log n)

# Approximate Hausdorff



- Assume we just want an algorithm that:
  - If $DH(A,B) \leq r$, answers YES
  - If $DH(A,B) \geq (1+ \varepsilon)r$, answers NO
- Algorithm:
  - Impose a grid with cell diameter $\varepsilon r$
  - For each $b \in B$, mark all cells within distance $r$ from $b$
  - For each $a \in A$, check if $a$'s cell is marked
- Time: $O(n/\varepsilon^2)$

# Alignment

- In general, A and B are not **aligned**
- So, in general, we want
  - $DH_T(A,B)=\min_{t\in T} DH(t(A),B)$ , where
    - T=translations
    - T=translations and rotations
    - …
  - Same for H
- How can we compute it ?

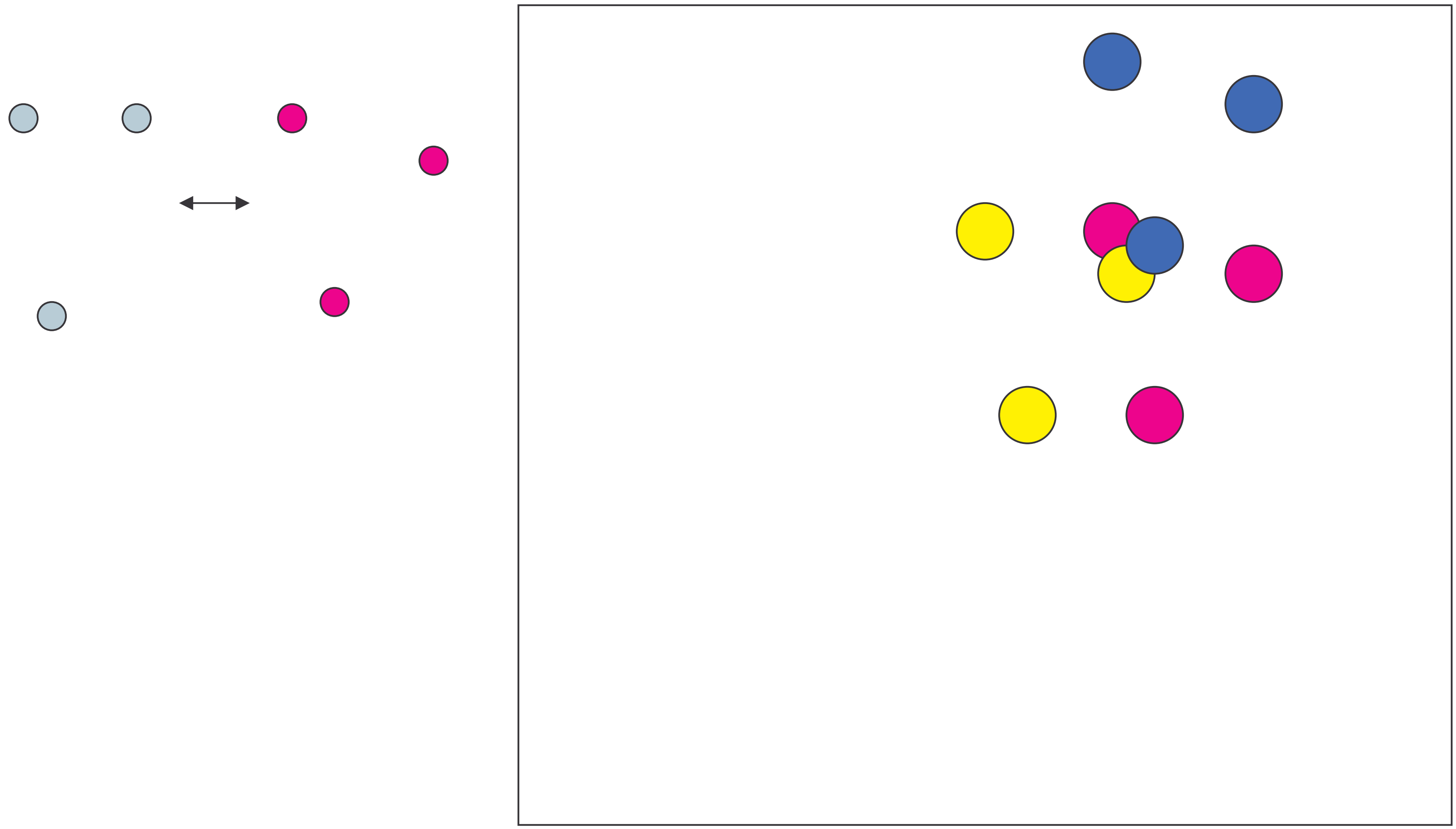
# Decision Problem

- Again, focus on if $DH_T(A,B) \leq r$
- For $a \in A$, define

$$T(a)=\{\ t:\ \exists b \in B\ \|t(a)-b\| \leq r\ \}$$

- $DH_T(A,B) \leq r$ iff $\cap_{a \in A} T(a)$ is non-empty

# Example



# Algorithm

- Compute the arrangement of all disks
- Check for non-emptiness
- Analysis:
  - Number of disks: $n^2$
  - Size of the arrangement: $O(n^4)$
  - Can compute the arrangement in time $O(n^4)$
  - Total time: $O(n^4)$
- Can be improved to $O(n^3 \log n)$ [Chew,Goodrich,Huttenlocher,Kedem,Kleinberg, Kravets'93]

# Running time

- Running time pretty high
- Can we speed it up?
- A $(1+\varepsilon)$-approximate algorithm for $DH_T$:
  - Impose a grid will pixel diameter $\varepsilon r$
  - Approximate each disk by $O(1/\varepsilon^2)$ cells
  - Each $T(a)$ is approximated by $O(n/\varepsilon^2)$ cells
  - Need to check intersection of $n$ $T(a)$'s
  - $O(n^2/\varepsilon^2)$ time

# Approximate Algorithm for H()

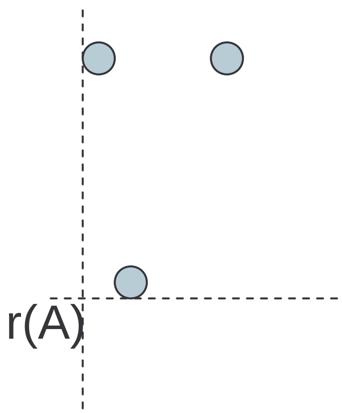
- Reference point: a point r(A) such that if we set t=r(B)-r(A) , then

$$H(\ t(A),\ B\ ) \leq c\ H_T(A,B)$$

- Note that t transforms r(A) into r(B)
- This will give us a c-approximate algorithm with time O(n log n)

# Constructing a Ref Point

- $r(A)=(\min_{a\in A} a_x, \min_{a\in A} a_y)$




- Introduced in [Alt, Behrends, Blomer'91]
- Improved in [Aichholzer, Alt, Rote'94]

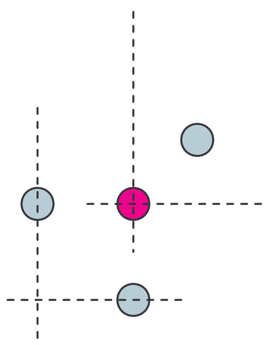
# Theorem

Theorem: $r(A)$ is a reference point with $c=1+\sqrt{2}$

Proof:

- Consider $t$ such that $H(t(A),B)=H_T(A,B)$
- What is $|\min_{a\in A} t(a)_x - \min_{b\in B} b_x|$ ?
- It is at most $H_T(A,B)$
- Same for $|\min_{a\in A} t(a)_y - \min_{b\in B} b_y|$
- Thus, we can translate $t(A)$ by a vector of length $\sqrt{2}\, H_T(A,B)$ , so that its reference point is aligned with $r(B)$

# The (1+$\sqrt{2}$) Factor is Tight (for this reference point)



# Exact Matching

- Check if $DH_T(A,B)=0$
  - Can modify the algorithm to allow small error
  - Technique used in practice

# Algorithm

- Take any pair $a,a' \in A$, let $r=||a-a'||$
- Find all pairs $b,b' \in B$ such that $||b-b'||=r$
- For all such pairs
  – Compute $t$ that transforms $(a,a')$ into $(b,b')$
  – Check if $t(A) \subseteq B$

# Analysis

- Choosing a,a' : constant time
- Enumerating b,b' : $O(n^2)$ time
- Checking match: $O(n\ P)$ , where $P$ is the number of pairs $b,b' \in B$ s.t. $||b-b'||=r$
- The bound for $P$ is …
- …deferred to the next episode J