Conveyer-Belt Alphabet

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Mathematics is often pursued purely for mathematical outcomes: theorems, proofs, open problems, conjectures. The open problems beg to be answered, and the quest to solve them is tantalizing and difficult—sometimes provably impossible. When successful, solving a problem is extremely rewarding, because mathematical proof offers (a small part of) the ultimate truth. But the pursuit toward this truth is equally interesting, and the field becomes richer if we allow research on mathematical problems to produce all sorts of outcomes, from art to puzzles to design. By exploring these connections between mathematics and diverse fields outside science, we see a new side to the original problems, leading to inspiration and, hopefully, solution.

The first two authors have found this open-ended approach to be both productive and enjoyable. Some examples are mathematical results in hinged dissection [DDEFF05, DDLS05] that later inspired a new mathematical font design [DD03] and an interactive sculpture [DDP06]; study of pleated origami that led to algorithmic sculpture [DDL99] and just recently culminated in a mathematical surprise that the objects we worked with do not in fact exist as we thought them to [DDHPT08]; study of curved-crease origami that led to sculpture at MoMA [DD08a] and architecturally relevant designs [KDD08]; pursuit of open problems in computational origami that remain unsolved but have led to a series of puzzle designs [DD08b].

Here we describe one such open-ended exploration, on a mathematical problem of wrapping an elastic loop around given wheels, and the mathematics, font design, and puzzles that resulted.

Conveyer-Belt Problem

Suppose we are given several disjoint disks pinned at their centers in the two-dimensional plane, and a closed elastic band, as in Figure 1. We think of the disk as a wheel that can spin freely around its center (but cannot otherwise move), and the band as a conveyer belt or rubber band, modeled as a stretchable closed loop in the plane that, at rest, tries to contract its length. Now we are asked to wrap the band around the disks in such a way that (1) the band touches every disk and (2) the band is taut, unable to contract in length given the disk obstacles. (Refer to Figure 2.) Equivalently, we want the band to wrap the disks so that rolling the band like a conveyer belt simultaneously turns all of the disks like wheels. More specifically, if the band rolls clockwise in the plane, then disks interior to the belt will rotate clockwise, while disks exterior to the belt will rotate counterclockwise.

The central mathematical question here is "what arrangements of disks have this kind of proper band wrapping?" Are there simple characterizations of when it is possible, or an efficient algorithm to tell whether given disks have a proper band wrapping? This seemingly simple geometrical

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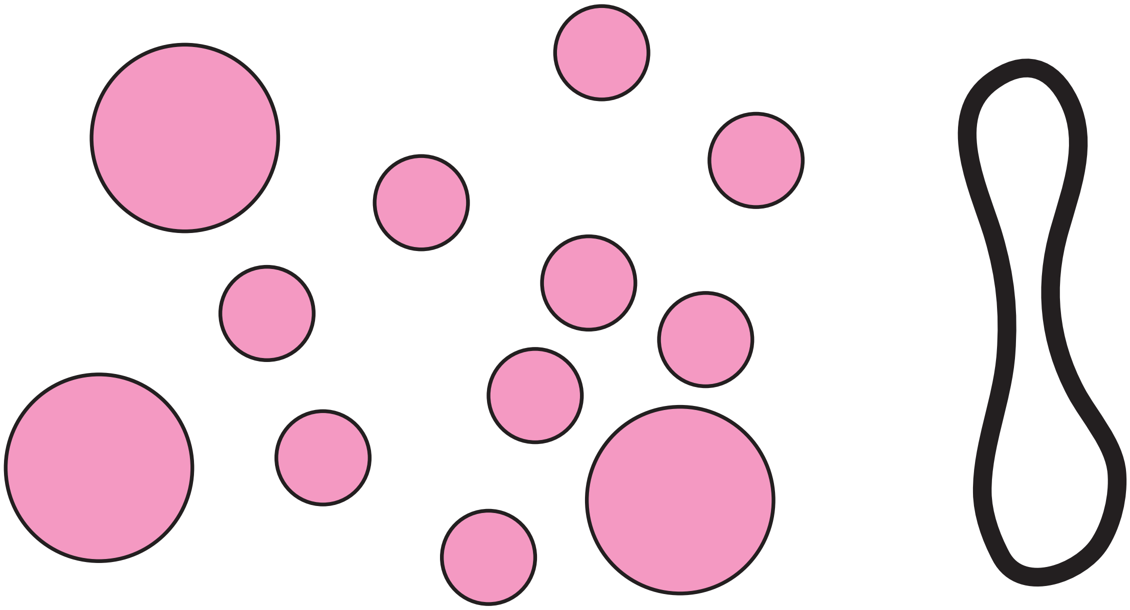

Figure 1: The input: Disks and an elastic band.

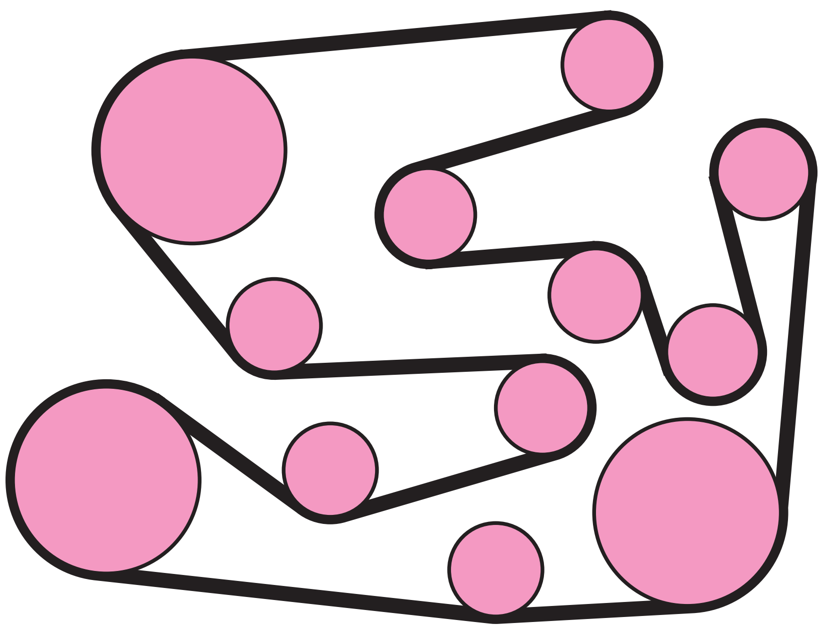

Figure 2: The goal: A valid wrapping of the elastic band around the disks.

problem has been posed by computational geometer Manuel Abellanas at several workshops, the earliest being the 1st Taller de Geometría Computacional in Cercedilla, Spain (2001), where the third author learned of the problem, and the most recent being the Workshop on Computational Geometry in Girona, Spain (2006), where the first author learned of the problem. The problem also recently appeared in print [Abe08]. By now dozens if not hundreds of researchers know of the problem, yet still surprisingly little is known.

The only result so far is that a proper band wrapping does not always exist. Javier Tejel and

Alfredo García found the seven-disk example in Figure 3. Note that the disks have vastly different sizes.

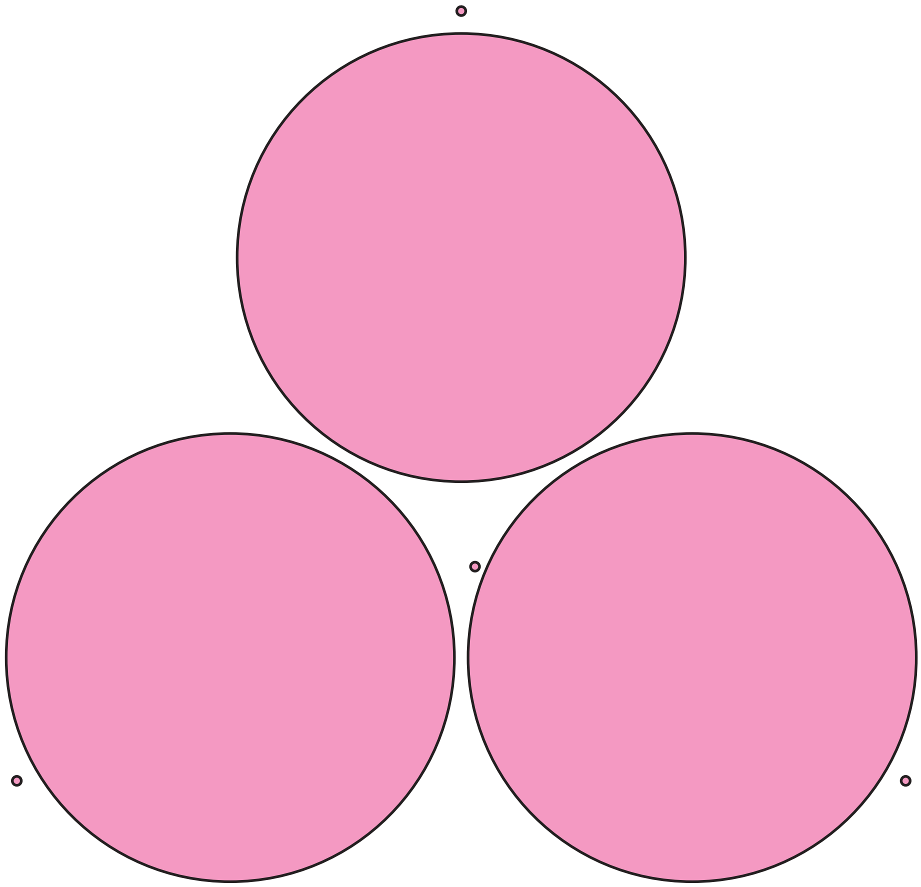

Figure 3: Seven disks of different sizes with no valid wrapping. [Javier Tejel and Alfredo García]

This example led Manuel Abellanas to pose the following more specific version of the conveyer-belt problem: do equal-size disks always have a proper band wrapping? This question has tantalized many researchers, and many attempts have been made to prove that the answer is "yes", though so far all have failed. Still, most conjecture that the answer is "yes", despite our lack of algorithm or proof.

Mathematical Quest

The authors met at M.I.T. in 2007 to discuss the conveyer-belt problem. We followed a common technique in mathematics of exploring special cases or easier variations to make partial steps toward a larger solution to the whole problem. We invented two interesting variations to the problem and made partial progress on each.

Our first variation allows the addition of extra ("Steiner") disks. How many disks do we need to add to a given arrangement of disks to guarantee that all disks together have a valid band wrapping? It is relatively easy that, given n disks, we might need to add at least $\Theta(n)$ disks: just repeat an example like Figure 3 $\Theta(n)$ times. What would be interesting is if this is roughly the worst case:

Conjecture 1 *Every arrangement of n disks can be augmented by $\Theta(n)$ additional disks so that the resulting arrangement has a valid band wrapping.*

We make this conjecture because the following algorithmic approach seems promising, though we have not yet been able to formalize it into a solution. Start from a traveling salesman tour, a closed path that visits each disk exactly once, but may not be taut. One way to compute such a tour is to compute the visibility graph of the given disks (which disks can see each other, unobscured by other disks), compute a spanning tree of that graph (a minimal set of visibility connections that form a connected network), and take an Euler tour of this tree (walking around the tree) but avoid revisiting a disk by taking detours around it. Now the idea is to turn this tour into a valid band wrapping by adding tiny disks at key locations to effectively navigate the band where desired. The details of this process remain vague, but they seem feasible.

The bigger challenge is to determine how few disks can be added. We can afford to add a few disks for every turn in the tour, but the worry would be that navigating tight gaps between disks could require more than just a few disks. Nonetheless, we believe that this challenge is surmountable, at least for some tour. The conjecture should also be easier to prove for equal-size disks. At worst, it should be relatively easy to find an algorithm adding $\Theta(n^2)$ disks, because navigating a gap among n disks should require at most $\Theta(n)$ added disks, and the tour requires only $\Theta(n)$ such navigations.

Our second variation is to suppose that all disks are both the same size and "separated", for an appropriate definition of separation. The idea for such a definition is to require that every two disks can see each other (in some sense) without unobstruction by other disks. In the weakest form, we can require just that there is a visibility line between the two disks; or we can require that every point on half of one disk can see some point on the other disk; or, in the strongest form, we can require that every point on half of one disk can see every point on half of the other disk. In these situations, or similar restrictions, it may be possible to follow a tour without adding any extra disks. Indeed, Manuel Abellanas has shown this result for the strongest of the three definitions of separation, and for any tour. This gives a positive solution to a special case of the equal-size conveyer-belt problem.

In the long term, our plan for this second variation is to pursue a hierarchical solution to the equal-size conveyer-belt problem, whereby we treat tightly packed clusters of disks separately from separated clusters of disks. We feel that these two extremes pose different challenges, each surmountable by themselves, hopefully in a way that can be combined. With separated clusters of disks, we can hope to use a solution to our second variation. With tightly packed clusters of disks, there are rather severe restrictions on how equal-size disks can be placed, and our hope is to exploit these constraints to find wrapping algorithms. This third variation is rather vague and probably the most difficult step in our plan.

Puzzle and Font Design

Having made some (but not a lot of) mathematical progress, we turned to challenging each other with puzzles along the lines of the conveyer-belt problem. The puzzle designer stuck identical push pins (representing equal-size disks) into a cork board, and the puzzle solver had to wrap a rubber band validly around the pins. To make the puzzle more challenging, however, the solution had to have one additional property: that it formed an English letter or word. The puzzle solver did not know which letter to aim for; the designer of course had one in mind, and aimed to ensure that the answer was unique. Figure 4 shows a few examples.

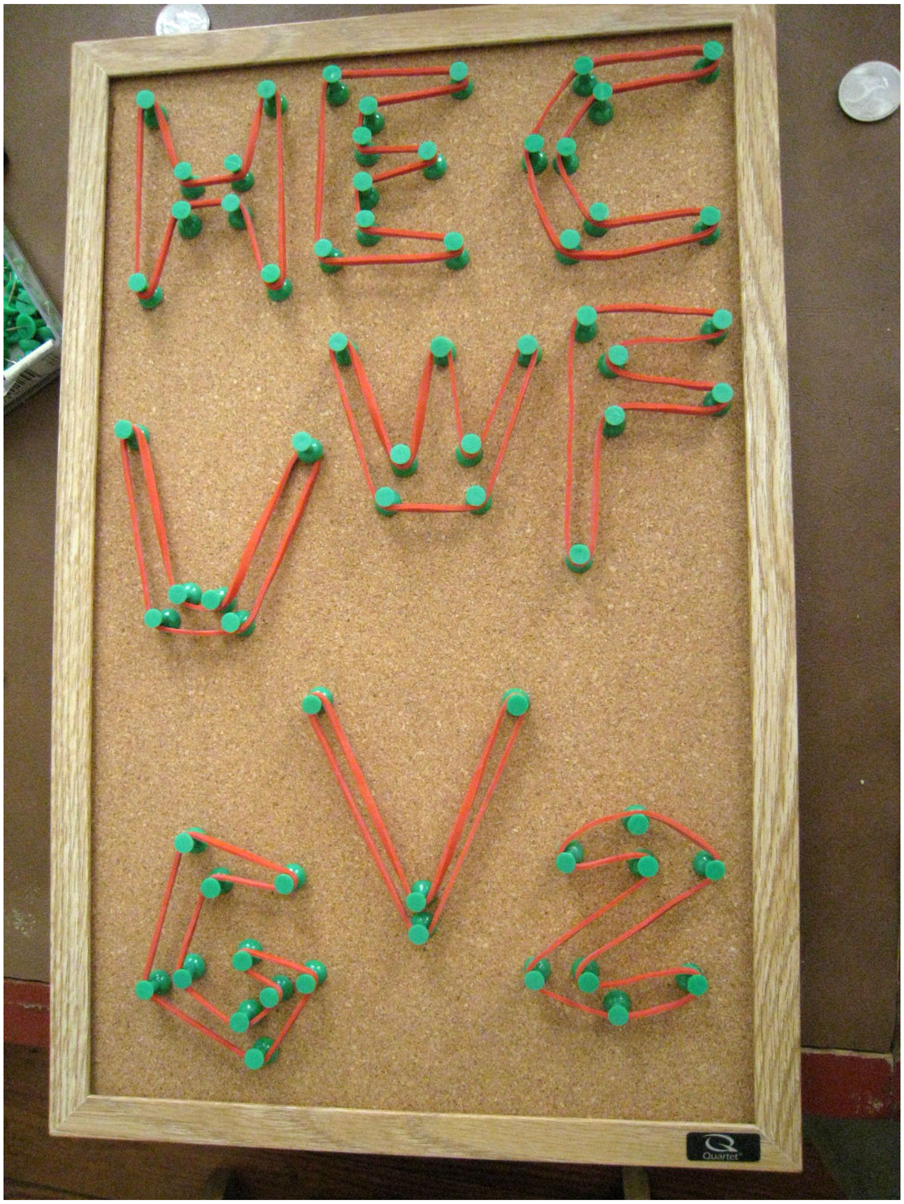

Figure 4: Some letters of the alphabet made with push pins and rubber bands.

This game quickly led to a series of puzzles and designs for making every letter and digit of the English alphabet. Figure 5 shows our preferred designs. The solved designs with the bands can be used as a new font with a mathematical backstory, as we have in the title of this paper. Alternatively, the puzzle designs without the bands can be used as a "secret code" that is most easily readable by those familiar with the mathematical problem. Figure 6 provides a simple coded puzzle for the reader.

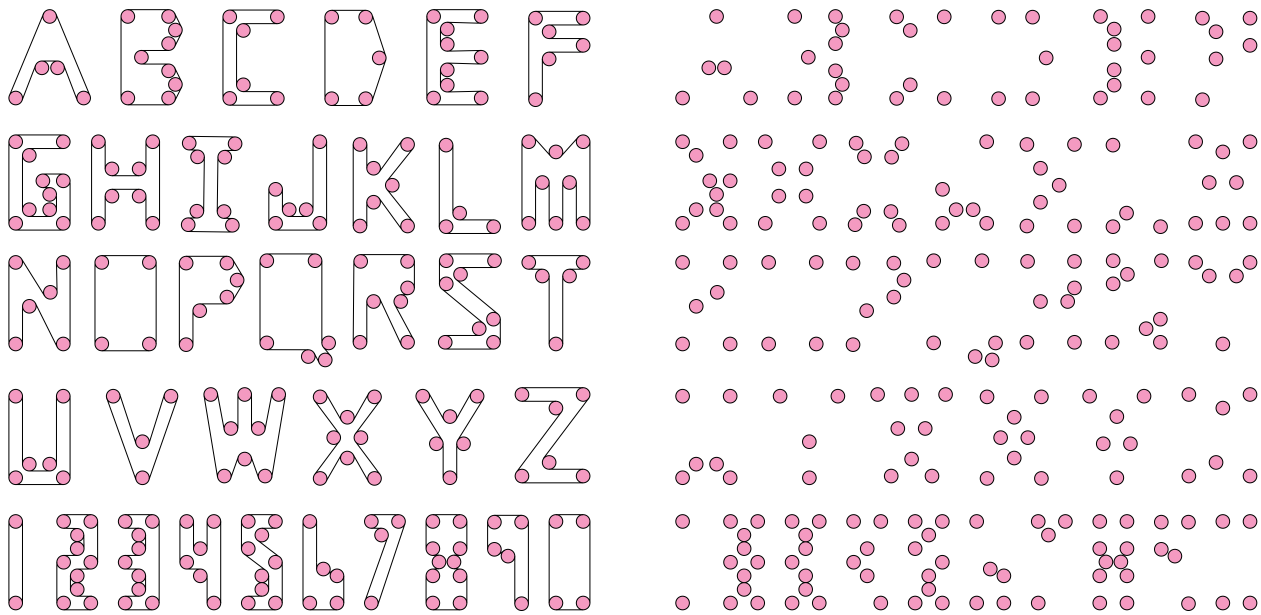

Figure 5: Our conveyer-belt alphabet, and the underlying disks.

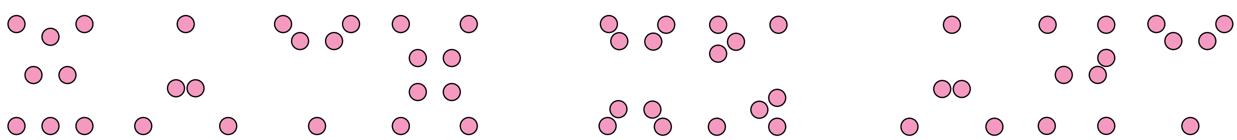

Figure 6: Concluding remark.

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Biographies

Erik D. Demaine is an Associate Professor in computer science at the Massachusetts Institute of Technology. His research interests range throughout algorithms, from data structures for improving web searches to the geometry of understanding how proteins fold to the computational difficulty of playing games. He received a MacArthur Fellowship (2003) as a "computational geometer tackling and solving difficult problems related to folding and bending—moving readily between the theoretical and the playful, with a keen eye to revealing the former in the latter". He recently published a book about folding, together with Joseph O'Rourke, called Geometric Folding Algorithms: Linkages, Origami, Polyhedra (Cambridge University Press, 2007). Inspired by his father, he also enjoys exploring the connections between mathematics and art.

Martin L. Demaine is an Artist-in-Residence and Visiting Scientist in computer science at the Massachusetts Institute of Technology. His work crosses the borders of art and science, ranging from mathematical geometry to sculpture in paper, glass, and recycled materials. His sculpture is in permanent collections around the world, from the Museum of Modern Art in New York and the National Gallery of Canada to the South Australian Museum in Adelaide. Together with Erik Demaine, he co-edited Tribute to a Mathemagician (2004, with Barry Cipra and Tom Rodgers) and A Lifetime of Puzzles (2008, with Tom Rodgers) in honor of the influential mathemagician Martin Gardner.

Belén Palop is an Associate Professor in the Computer Science Department at the University of Valladolid in Spain. Her main area of research is computational geometry. More specifically, she has directed her attention in recent years to Voronoi diagrams, one of the most natural geometrical structures, able to describe processes from forestry to crystallography or demography.