Solitaire Clobber

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Abstract. Clobber is a new two-player board game. In this paper, we introduce the 1-player variant Solitaire Clobber where the goal is to remove as many stones as possible from the board by alternating white and black moves. We show that a checkerboard configuration on a single row (or single column) can be reduced to about $n/4$ stones. For boards with at least two rows and columns, we show that a checkerboard configuration can be reduced to a single stone if and only if the number of stones is not a multiple of three, and otherwise it can be reduced to two stones. But in general it is NP-complete to decide whether an arbitrary Clobber configuration can be reduced to a single stone.

1 Introduction

Clobber is a new two-player combinatorial board game with complete information, recently introduced by Albert, Grossman, and Nowakowski (see [5]). It is played with black and white stones occupying some subset of the squares of an $n \times m$ checkerboard. The two players, White and Black, move alternately by picking up one of their own stones and *clobbering* an opponent's stone on a horizontally or vertically adjacent square. The clobbered stone is removed from the board and replaced by the stone that was moved. The game ends when one player, on their turn, is unable to move, and then that player loses.

We say a stone is *matching* if it has the same color as the square it occupies on the underlying checkerboard; otherwise it is *clashing*. In a *checkerboard configuration*, all stones are matching, i.e., the white stones occupy white squares and the black stones occupy black squares. And in a *rectangular* configuration, the stones occupy exactly the squares of some rectangular region on the board. Usually, Clobber starts from a rectangular checkerboard configuration, and White moves first (if the total number of stones is odd we assume that it is White who has one stone less than Black).

At the recent Dagstuhl Seminar on Algorithmic Combinatorial Game Theory [1], the game was first introduced to a broader audience. Tomáš Tichý from Prague won the first Clobber tournament, played on a 5×6 board, beating his supervisor Jiří Sgall in the finals. Not much is known about Clobber strategies, even for small boards, and the computation of CGT game values is also only in its preliminary stages.

In this paper we introduce *Solitaire Clobber*, where a single player (or two cooperative players) tries to remove as many stones as possible from the board by alternating white and black moves. If the configuration ends up with k immovable stones, we say that the initial board configuration is *reduced* to k stones, or k-*reduced*. Obviously, 1-reducibility can only be possible if half of the stones are white (rounded down), and half of the stones are black (rounded up). But even then it might not be possible.

We prove the following necessary condition for a Clobber position to be 1-reducible: The number of stones plus the number of clashing stones cannot be a multiple of three. Surprisingly, this condition is also sufficient for truly two-dimensional rectangular checkerboard configurations (i.e., with at least two rows and two columns). And if the condition is not true, then the board is 2-reducible (with the last two stones separated by a single empty square), which is the next-best possible. A similar 2-coloring argument can be used to solve Question 3 of the 34th International Mathematical Olympiad 1993 [3] which asked to prove that the peg solitaire game (a peg can jump over an adjacent peg onto an empty square, and the jumped over peg is removed) on an $n \times n$ grid can be reduced to a single peg when $n \equiv 1 \bmod 3$. However, in general, we show that it is NP-complete to decide whether an arbitrary non-rectangular non-checkerboard configuration is 1-reducible.

If we play one-dimensional Solitaire Clobber (i.e., the board consists of a single row of stones) reducibility is more difficult. We show that the checkerboard configuration can be reduced to $\lceil n/4 \rceil + \{1 \text{ if } n \equiv 3 \pmod 4\}$ stones, no matter who moves first, and that this bound is best possible even if we do not have to alternate between white and black moves. This result was obtained independently by Grossman [2].

This paper is organized as follows. In Section 2, we analyze the reducibility of checkerboard configurations on a line. In Section 3, we study reducibility of two-dimensional rectangular checkerboard configurations. And in Section 4 we show that deciding 1-reducibility is NP-complete in general. We conclude with some open problems in Section 5.

2 One-Dimensional Solitaire Clobber

In this section we study Solitaire Clobber played on a board consisting of a single row of stones. Let A_n denote the checkerboard configuration, i.e., an alternating sequence of white and black stones. By symmetry, we can assume throughout this section that A_n always starts with a black stone, so we have $A_n = \bullet\circ\bullet\circ\cdots$. We first show an upper bound on the k-reducibility of checkerboard configurations.

Theorem 1. *However, in general, we show that it is NP-complete to decide whether an arbitrary non-rectangular non-checkerboard configuration is 1-reducible.*

If we play one-dimensional Solitaire Clobber (i.e., the board consists of a single row of stones) reducibility is more difficult. We show that the checkerboard configuration can be reduced to $\lceil n/4 \rceil + \{1 \text{ if } n \equiv 3 \pmod 4\}$ *stones, no matter* *who moves first, and that this bound is best possible even if we do not have to alternate between white and black moves. This result was obtained independently by Grossman [2].*

For $n \geq 1$, the configuration A_n can be reduced to $\lceil n/4 \rceil + \{1$ if $n \equiv 3 \pmod 4\}$ stones by an alternating sequence of moves, no matter who is to move first.

Proof. Split the configuration A_n into $\lceil n/4 \rceil$ substrings, all but possibly one of length four. Each substring of length one, two, or four can be reduced to one stone by alternating moves, no matter which color moves first. And a substring of size three can be reduced to two stones by one move, no matter which color moves first. □

In this move sequence, we end up with one isolated stone somewhere in the middle of each block of four consecutive stones. One might wonder whether a more clever strategy could end up with one stone at the end of each subblock, and then we could clobber one more stone in each pair of adjacent stones from the subblocks. Unfortunately, this is not possible, as shown by the following matching lower bound. The lower bound holds even if we are not forced to alternate between white and black moves. We give a simple proof for the theorem due to Grossman [2].

Theorem 2. *Let $n \geq 1$. Even if we are not restricted to alternating white and black moves, the configuration A_n cannot be reduced to fewer than $\lceil n/4 \rceil + \{1$ if $n \equiv 3 \pmod 4\}$ stones.*

Proof. First, it is not possible to reduce A_3 or A_5 to a single stone. Second, each stone in the final configuration comes from some contiguous substring of stones in the initial configuration. But each of these substrings can have only one, two, or four stones. Thus, there are at least $\lceil n/4 \rceil$ stones left at the end, and even one more if $n \equiv 3 \pmod 4$. □

Somewhat surprisingly, the tight bound of Theorems 1 and 2 is not monotone in n, the number of stones in the initial configuration. See Table 1.

3 Rectangular Solitaire Clobber

In this section we study reducibility of rectangular checkerboard configurations with at least two rows and two columns. We first show a general lower bound on the reducibility that holds for arbitrary Clobber configurations. For a configuration C, we denote the quantity "number of stones plus number of clashing stones" by $\delta(C)$.

As it turns out, $\delta(C) \pmod 3$ actually divides all clobber configurations into three equivalence classes. Any configuration will stay in the same equivalence class, after any number of moves. Because one of the three equivalence classes (with $\delta(C) \equiv 0 \pmod 3$) does not contain configurations with a single stone, all configurations in this equivalence class are not 1-reducible. As in the 1-dimensional lower bound of Theorem 2, this is true even if we allow arbitrary non-alternating move sequences.

Configuration		Reducibility
A_1	●	1
A_2	●○	1
A_3	●○●	2
A_4	●○●○	1
A_5	●○●○●	2
A_6	●○●○●○	2
A_7	●○●○●○●	3
A_8	●○●○●○●○	2
A_9	●○●○●○●○●	3
A_{10}	●○●○●○●○●○	3
A_{11}	●○●○●○●○●○●	4
A_{12}	●○●○●○●○●○●○	3

Table 1. Reducibility of one-dimensional checkerboard Clobber configurations.

Theorem 3. *For a configuration C, $\delta(C) \pmod 3$ does not change after an arbitrary move sequence.*

Proof. If we move a matching stone in C then δ drops by one because we clobber another matching stone, and δ rises by one because our stone becomes clashing, so δ actually does not change in this move. If we move a clashing stone then δ drops by two because we clobber another clashing stone, and δ drops by another one because our stone becomes matching, resulting in a total drop of three for the move. □

Corollary 1. *A configuration C with $\delta(C) \equiv 0 \pmod 3$ is not 1-reducible.*

Proof. A single stone can only have δ equal to one or two (depending on whether it is a matching or clashing stone). Thus, by the previous theorem, configurations C with $\delta(C) \equiv 0 \pmod 3$ are not 1-reducible. □

The rest of this section is devoted to a proof that this bound is actually tight for rectangular checkerboard configurations:

Theorem 4. *For $n, m \geq 2$, a rectangular checkerboard configuration with n rows and m columns is 2-reducible if $nm \equiv 0 \pmod 3$, and 1-reducible otherwise.*

We present an algorithm that computes a sequence of moves that reduces the given checkerboard configuration to one or two stones as appropriate.

We distinguish cases in a somewhat complicated way. There are finitely many cases with $2 \leq n, m \leq 6$; these cases can be verified trivially, as shown in Appendix A. The remaining cases have at least one dimension with at least seven stones; by symmetry, we ensure that the configuration has at least seven columns. These cases we distinguish based on the parities of n and m:

- **Case EE:** Even number of rows and columns [Section 3.2]

- **Case OE:** Odd number of rows, even number of columns [Section 3.3]
- **Case EO:** Even number of rows, odd number of columns [Section 3.4]
- **Case OO:** Odd number of rows and columns [Section 3.4]

Cases OE and EO are symmetric for configurations with at least seven rows and at least seven columns. By convention, we handle such situations in Case EO. But when one dimension is smaller than seven, we require that dimension to be rows, forcing us into Case OE or Case EO and breaking the symmetry. In fact, we solve these instances of Case OE by rotating the board and solving the simpler cases E3 and E5 (even number of rows, and three or five columns, respectively).

Section 3.1 gives an overview of our general approach. Section 3.2 considers Case EE, which serves as a representative example of the procedure. Section 3.3 extends this reduction to Case OE (when the number of rows is less than seven), which is also straightforward. Finally, Section 3.4 considers the remaining more-tedious cases in which the number of columns is odd.

3.1 General Approach

In each case, we follow the same basic strategy. We eliminate the stones on the board from top to bottom, two rows at a time. More precisely, each *step* reduces the topmost two rows down to $O(1)$ stones (usually one or two) arranged in a fixed pattern that touches the rest of the configuration through the bottom row.

There are usually four types of steps, repeated in the order

$$(1),\ \underbrace{(2),\ (3),\ (4)},\ \underbrace{(2),\ (3),\ (4)},\ \underbrace{(2),\ (3),\ (4)},\ \ldots.$$

Step (1) leaves a small remainder of stones from the top two rows in a fixed pattern. Step (2) absorbs this remainder and the next two rows, in total reducing the top four rows down to a different pattern of remainder stones. Step (3) leaves yet another pattern of remainder stones from the top six rows. Finally, step (4) leaves the same pattern of remainder stones from step (1), so the sequence can repeat (2), (3), (4), (2), (3), (4), ...

In some simple cases, steps (1) and (2) leave the same pattern of remainder stones. Then just two types of steps suffice, repeating in the order (1), (2), (2), (2), ... In other cases, three steps suffice.

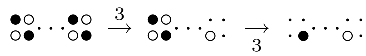
In any case, the step sequence may terminate with any type of step. Thus, we must also show how to reduce each pattern of remainder stones down to one or two stones as appropriate; when needed, these final reductions are enclosed by parentheses because they are only used at the very end. In addition, if the total number of rows is odd, the final step involves three rows instead of two rows, and must be treated specially.

In the description below, a single move is denoted by $\rightarrow$. But we often do not show long move sequences completely. Instead, we usually 'jump' several moves at a time, denoted by $\stackrel{a}{\rightarrow}$ or $\underset{a}{\rightarrow}$, depending on whether White or Black moves first, where a denotes the number of moves we jump.

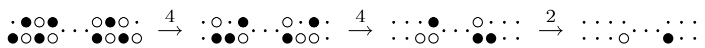
3.2 Case EE: Even Number of Rows and Columns

We begin with the case in which both n and m are even. This case is easier than the other cases: the details are fairly clean. It serves as a representative example of the general approach.

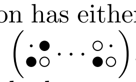
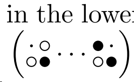
Because the number of columns is even and at least seven, it must be at least eight. Every step begins by reducing the two involved rows down to a small number of columns. First, we clobber a few stones to create the following configuration in which the lower row has two more stones than the upper row, one on each side:



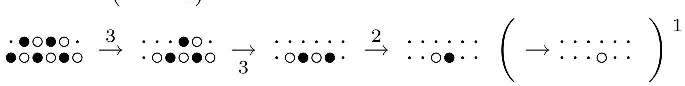
Then we repeatedly apply the following reduction, in each step removing six columns, three on each side:

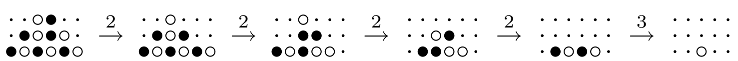


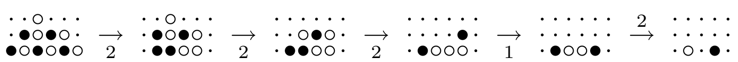
We stop applying this reduction when the bottom row has just six, eight, or ten columns left, and the top row has four, six, or eight columns, depending on whether $m \equiv 2$, 1, or 0 (mod 3), respectively.

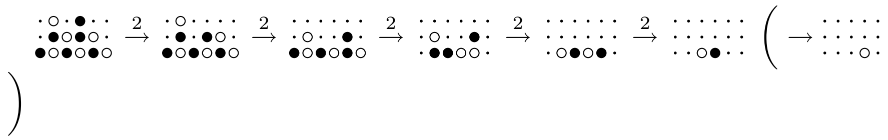
The resulting two-row configuration has either a black stone in the lower-left and a white stone in the lower-right , or vice versa . We show reductions for the former case; the latter case is symmetric.

Case 1: $m \equiv 2 \pmod 3$

(1) 

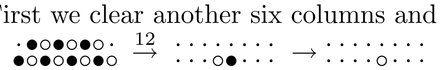
(2) 

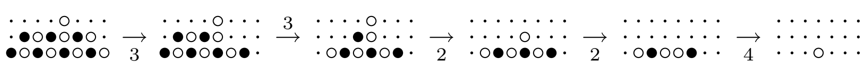
(3) 

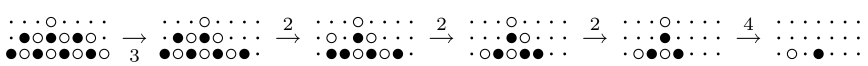
(4) 

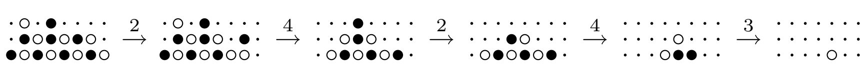
Case 2: $m \equiv 1 \pmod 3$

(1) First we clear another six columns and obtain



(2) 

(3) 

(4) 

[1] Parenthetical moves are made only if this is the final step.

Case 3: $m \equiv 0 \pmod 3$

(1) First we clear another six columns and obtain

```
     ·●○●○●○●○·  -12->  ····○●····  -4->  ··········
     ●○●○●○●○●○         ···○●○●···        ···○·●····
```

```
     ···○·●····         ···○·●····        ···○······        ··········        ··········
(2)  ·●○●○●○●○·  -6->   ···●○●○···  -4->  ···●●·○···  -4->  ··········  -4->  ··········
     ●○●○●○●○●○         ·○●○●○●○●·        ··○○●·●○●·        ··○●●·○○●·        ····○·●···
```

```
     ····○·●···         ····○·●···        ······●···        ··········        ··········
(3)  ·●○●○●○●○·  -6->   ···●○●○···  -4->  ····○●○···  -4->  ······○···  -4->  ··········
     ●○●○●○●○●○         ·○●○●○●○●·        ··○○●○●●··        ··○●·○●●··        ···○·●····
```

3.3 Case OE: Odd Number of Rows, Even Number of Columns

To extend Case EE from the previous section to handle an odd number of rows, we could provide extra termination cases with three instead of two rows for any step. Because these steps are always final, they may produce an arbitrary result configuration with one or two stones.

However, as observed before, we only need to consider configurations with three or five rows in Case OE (any other configuration can be rotated into a Case EO). It turns out that we can describe their reduction more easily (and conform with all other cases) by first rotating them. Thus, the following reductions use the general approach from Section 3.1 to reduce configurations with three or five columns and an even number of rows.

Three Columns:

```
(1)  ●○●  -2->  ··●  -2->  ···
     ○●○        ●○○        ○·●
```

```
     ○·●        ○··        ···        ···
(2)  ●○●  -2->  ●·●  -2->  ○·●  -2->  ···
     ○●○        ○●○        ●·○        ○·●
```

Five Columns:

```
(1)  ●○●○●  -2->  ·●●○●  -2->  ·●●●·  -2->  ·○●··  -3->  ·····
     ○●○●○        ·○○●○        ·○○○·        ··○●·        ··○··
```

```
     ··○··        ··○··        ·····        ·····
(2)  ●○●○●  --->  ·●●●·  --->  ·●○··  -3->  ·····
     ○●○●○   4    ·○○○·   3    ·○●··        ·○···
```

```
     ·○···        ·○···        ·····        ·····
(3)  ●○●○●  --->  ·●●●·  --->  ·○·●·  -2->  ·····
     ○●○●○   4    ·○○○·   3    ·●·○·        ·○·●·
```

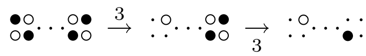
```
     ·○·●·        ·○···        ·····        ·····        ·····
(4)  ●○●○●  -4->  ·●●●○  -2->  ·●·●○  -2->  ···○·  -3->  ·····
     ○●○●○        ·○○●·        ·○○●·        ·●○●·        ··○··
```

3.4 Cases EO and OO: Odd Number of Columns

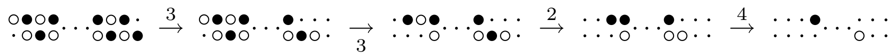
Finally we consider the case of an even or odd number of rows and an odd number of columns. For each step, we give two variants, one reduction from two rows and one reduction from three rows. The latter case is applied only at the end of the reduction, so it does not need to end with the same pattern of remainder stones. Also, for an odd number of rows, the initial symmetrical removal of columns from both ends of the rows in a step is done first for the final three-row step, before any other reduction; this order is necessary because the three-row symmetrical removal can start only with a White move.

The number of columns is at least seven. Every step begins by reducing the two or three involved rows down to a small number of columns.

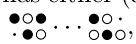
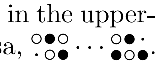
Two Rows. First, we clobber a few stones to create the following configuration in which the upper row has one more stone on the left side than the lower row, and the lower row has one more stone on the right side than the upper row:

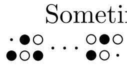


Similar to Case EE, we repeatedly apply the following reduction, in each step removing six columns, three on each side:

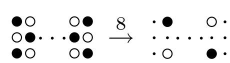


We stop applying this reduction when the total number of columns is just five, seven, or nine, so each row has four, six, or eight occupied columns, depending on whether $m \equiv 1$, 0, or 2 (mod 3), respectively.

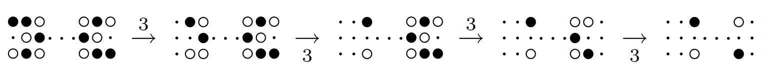
The resulting two-row configuration has either (a) a black stone in the upper-left and a white stone in the lower-right, , or (b) vice versa, . We will show reductions from both configurations. It turns out that configuration (a) is more difficult to handle because it is not always possible to end up with a single stone (or pair of stones) on the bottom row. In that case, we will make the last move parenthetical, omitting it whenever this step is not the last.

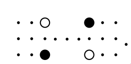
Sometimes we also need to start from the configuration (a′)  or (b′)  which are the mirror images of the configurations (a) and (b). These starting points can be achieved by applying the reductions above upside-down.

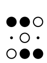
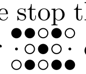
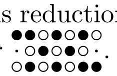
Three rows. First, we clobber a few stones to create the following configuration:



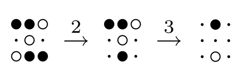
Then, we reduce long rows by four columns at a time (not six as in the two-row reductions):

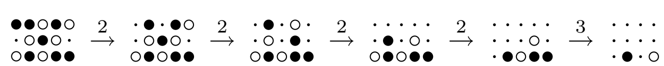


Note that we can also obtain the symmetric configuration . Because we cannot perform this reduction with Black starting, we must perform this reduction at the very beginning of the entire algorithm, before any other steps.

We stop this reduction when we have reached one of the three configurations  or  or . We are not able to reduce the last configuration further because in some cases it isolates the remaining stones from the rows above.

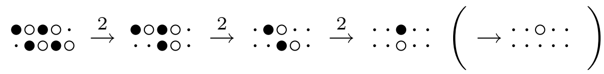
Reductions. Now we show how to reduce the configurations described above following the general approach from Section 3.1. For the case of three rows, we only need to consider the following two reductions in step (1):

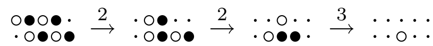
(1) 

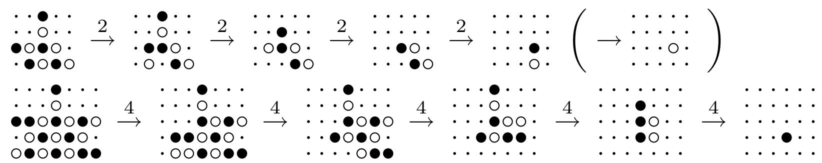
(1′) 

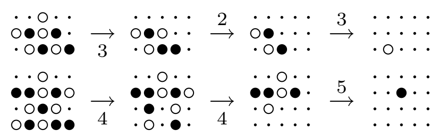
Case 1: $m \equiv 1 \pmod 3$

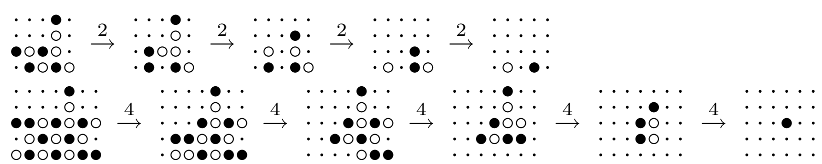
The initial configuration is of type (a) for $m = 13 + 12k$ columns and of type (b) for $m = 7 + 12k$ columns, for $k \geq 0$.

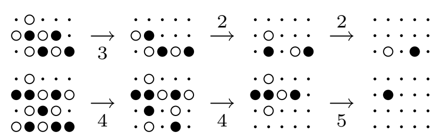
(1a) 

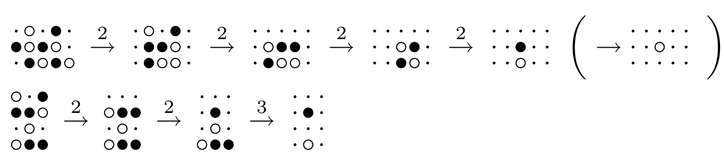
(1b) 

(2a) 

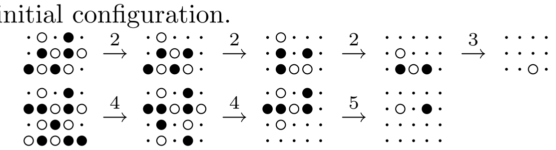
(2b) 

(3a) 

(3b) 

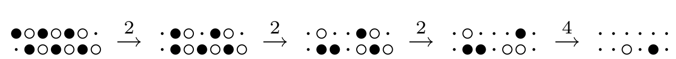
(4a) 

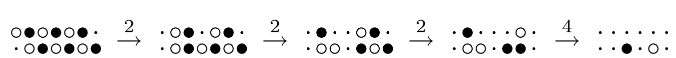
(4b′) We must reduce rows seven and eight starting with the mirrored standard initial configuration.

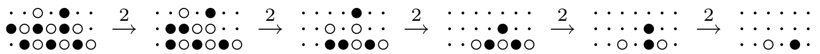


Case 2: $m \equiv 0 \pmod 3$

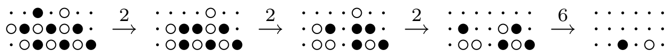
The initial configuration is of type (a) for $m = 15 + 12k$ columns and of type (b) for $m = 9 + 12k$ columns, for $k \geq 0$.

(1a) 

(1b) 

(2a) 

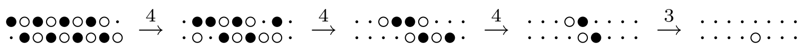
For three rows, this case is identical to Case 1(4b).

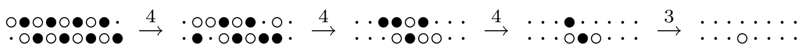
(2b) 

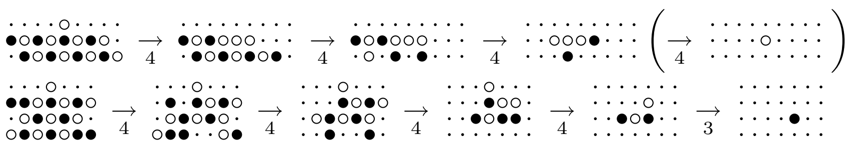
For three rows, this case is symmetric to (4a) in Case 1 (with the mirrored initial configuration).

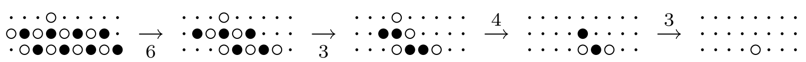
Case 3: $m \equiv 2 \pmod 3$

The initial configuration is of type (a) for $m = 17 + 12k$ columns and of type (b) for $m = 11 + 12k$ columns, for $k \geq 0$.

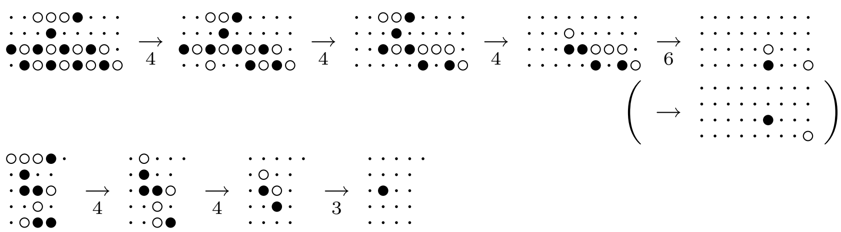
(1a) 

(1b) 

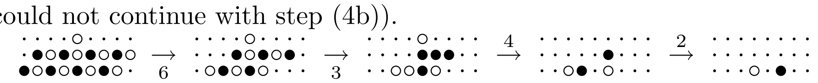
(2a) 

(2b) 

For three rows, this case is identical to Case 1(3b).

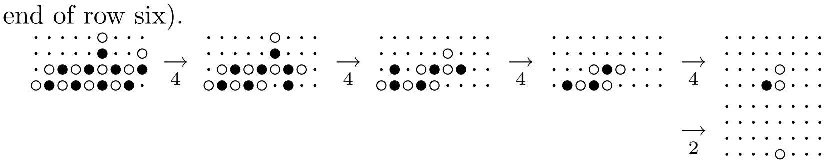
(3a) 

(3b) We must reduce rows five and six starting with the mirrored standard initial configuration (we could also solve the standard configuration, but then we could not continue with step (4b)).

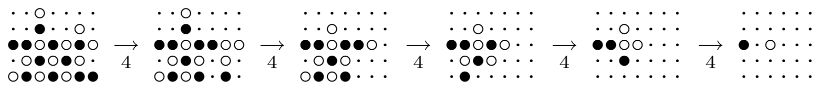


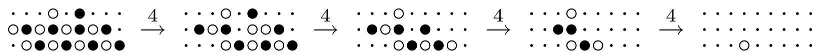
For three rows, this case is identical to Case 1(2b).

(4a) We must reduce rows seven and eight starting with the mirrored standard initial configuration (because of the white single stone left over at the right end of row six).



For three rows, we must reduce the number of columns a little bit asymmetrically (remove four additional columns on the left side) and then do the following reduction.



(4b) 

For three rows, this case is identical to Case 1(4b).

4 NP-Completeness of 1-Reducibility

In this section we consider arbitrary initial Clobber positions that do not need to have a rectangular shape or the alternating checkerboard placement of the stones. We show that then the following problem is NP-complete.

> **Problem** SOLITAIRE-CLOBBER**:**
> Given an arbitrary initial Clobber configuration, decide whether we can reduce it to a single stone.

The proof is by reduction from the Hamiltonian circuit problem in grid graphs. A *grid graph* is a finite graph embedded in the Euclidean plane such that the vertices have integer coordinates and two vertices are connected by an edge if and only if their Euclidean distance is equal to one.

> **Problem** GRID-HAMILTONICITY**:**
> Decide whether a given grid graph has a Hamiltonian circuit.

Itai *et al.* proved that GRID-HAMILTONICITY is NP-complete [4, Theorem 2.1].

Theorem 5. SOLITAIRE-CLOBBER *is NP-complete.*

Proof. We first observe that SOLITAIRE-CLOBBER is indeed in NP, because we can easily check in polynomial time whether a proposed solution (which must have only $n-1$ moves) reduces the given initial configuration to a single stone.

We prove the NP-completeness by reduction from GRID-HAMILTONICITY. Let G be an arbitrary grid graph with n nodes, embedded in the Euclidean plane. Let v be a node of G with maximum y-coordinate, and among all such nodes the node with maximum x-coordinate. If v does not have a neighbor to the left then G cannot have a Hamiltonian circuit. So assume there is a left neighbor w of v. Note that v has degree two and therefore any Hamiltonian circuit in G must use the edge (v, w).

Then we construct the following Clobber configuration (see Figure 1). We put a black stone on each node of G. We place a single white stone just above w, the *bomb*. We place a vertical chain of n white stones above v, the *fuse*, and another single black stone, the *fire*, on top of the fuse. Altogether we have placed $n+1$ white and $n+1$ black stones, so this is a legal Clobber configuration.

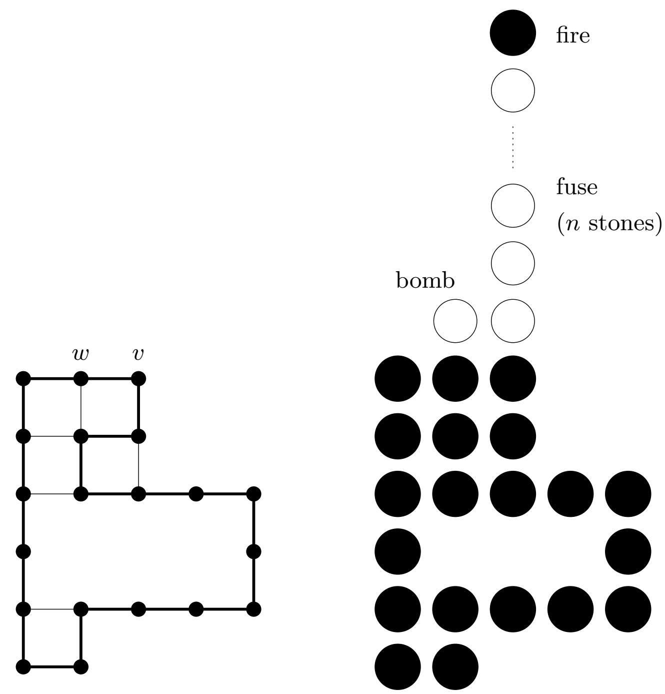


Fig. 1. An n-node grid graph with Hamiltonian circuit and the corresponding Clobber configuration that can be reduced to a single stone.

If G has a Hamiltonian circuit C then the bomb can clobber all black nodes of G, following C starting in w and ending in v after n rounds. At the same time, the black fire can clobber the n stones of the fuse and end up just above v after n rounds. But then in a last step the bomb can clobber the fire, leaving a single stone on the board.

On the other hand, if the initial configuration can be reduced to a single stone then White cannot move any stone on the fuse (because that would disconnect the black fire from the stones on G), so it must move the bomb until Black has clobbered the fuse. But that takes n steps, so White must in the meanwhile clobber all n black stones of G, that is, it must walk along a Hamiltonian circuit in G. $\square$

5 Conclusions

We have seen that reducing to the minimum number of stones is polynomially solvable for checkerboard rectangular configurations, and is NP-hard for general configurations. What about checkerboard non-rectangular configurations and rectangular non-checkerboard configurations?

We have also seen a lower bound on the number of stones to which a configuration can be reduced that is based on the number of stones plus the number of stones on squares of different color. It would be interesting to identify other structural parameters of a configuration that influence reducibility.

References

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2. J. P. Grossmann, March 2002. Private communication.
3. 34th International Mathematical Olympiad, Istanbul, Turkey, Question 3, 1993. See `http://www.kalva.demon.co.uk/imo/imo93.html`.
4. A. Itai, C. H. Papadimitriou, and J. L. Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal on Computing*, 11(4):676–686, 1982.
5. D. Wolfe. Clobber Research, 2002. `http://www.gac.edu/~wolfe/games/clobber`.

A Small Cases

Our proof of Theorem 4 requires us to verify reducibility for all instances with $2 \leq n, m \leq 6$. This fact can be checked easily by a computer, but for completeness we give the reductions here. By symmetry, we only need to show the cases with $n \leq m$. Reductions of 2×3, 2×5, 3×4, 3×6, 4×5, and 5×6 boards are already given in Section 3.3. Eight more small boards remain. We assume White moves first.

2×2:

●○	$\rightarrow$	●·	$\rightarrow$	··	$\rightarrow$	··
○●		○○		●○		○·

2×4:

●○●○	$\rightarrow$	●○●·	$\rightarrow$	●○··	$\rightarrow$	●○··	$\rightarrow$	·○··	$\rightarrow$	····	$\rightarrow$	····	$\rightarrow$	····
○●○●		○●○○		○●●○		○●○·		●●○·		●○○·		·●○·		·○··

2×6:

●○●○●○	$\xrightarrow{3}$	●○●○··	$\xrightarrow[3]{}$	··●○··	$\xrightarrow{2}$	······	$\xrightarrow{2}$	······
○●○●○●		○●○●○·		·●○●○·		·●●○○·		··●·○·

3×3:

●○●		○·●		···
○●○	$\xrightarrow{2}$	●·○	$\xrightarrow{3}$	···
●○●		●○●		○·●

3×5:

●○●○●		●○●○●		·●●○●		··●○●		···●○		···○·		·····
○●○●○	$\xrightarrow{2}$	··○●○	$\xrightarrow{2}$	··○●○	$\xrightarrow{2}$	···●○	$\xrightarrow{2}$	···●·	$\xrightarrow{2}$	···●·	$\xrightarrow{3}$	·····
●○●○●		○●●○●		·○●○●		·○●○●		·○●○●		·○●●·		·●·○·

4×4: We reduce the upper two rows as in the 2×4 board, then Black moves next:

·○··		·○··		····		····		····		····		····		····		····
●○●○	$\rightarrow$	·●●○	$\rightarrow$	·○●○	$\rightarrow$	·●·○	$\rightarrow$	·●·○	$\rightarrow$	···○	$\rightarrow$	····	$\rightarrow$	····	$\rightarrow$	····
○●○●		○●○●		○●○●		○●○●		·○○●		·●○●		·●○○		··●○		··○·

4×6: We reduce the upper two rows as in the 2×6 board, then White moves next:

··●·○·		····○·		······		······		······		······
●○●○●○	$\xrightarrow{2}$	●·●○●○	$\xrightarrow{2}$	●·●○○●	$\xrightarrow{2}$	●·●○●·	$\xrightarrow{2}$	●··●○·	$\xrightarrow{4}$	······
○●○●○●		○●○●○●		○●○●○·		○○·●○·		○○·●··		·●·○··

6×6: We reduce the upper four rows as in the 4×6 board, then White moves next:

·●·○··		···○··		···○··		······		······		······		······
●○●○●○	$\xrightarrow{2}$	○●●○●○	$\xrightarrow{2}$	·○●●●○	$\xrightarrow{2}$	·●●○●○	$\xrightarrow{2}$	··●·●○	$\xrightarrow{2}$	··●···	$\xrightarrow{2}$	······
○●○●○●		·●○●○●		·●○·○●		··○·○●		··○·○●		··○·●○		··●·○·