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Chess Equilibrium Puzzles

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Chess is perhaps the best-known two-player board game, played on a square chessboard consisting of 64 smaller squares, alternating in black and white colors, and 16 pieces for each player. While chess itself is a game that is enjoyed internationally by a wide variety of people, chess *puzzles* are also enjoyed by both chess professionals and chess enthusiasts alike. Many professional puzzle hunts feature some form of chess puzzles, usually requiring puzzle solvers to determine previous moves or a move that would mate in 1. Some examples of such puzzles may be found in Martin Gardner's *The Colossal Book of Short Puzzles and Problems* [**4**]. In this short exposition, we introduce a new type of chess puzzle that we call *Chess Equilibrium* puzzles. Our puzzles are a new spin on the traditional retrograde or mate-in-1 puzzles found in recreational mathematics.

Chess Equilibrium Rules Chess Equilibrium is a new type of puzzle that challenges the puzzle solver to place a set of required pieces on an $n \times n$ chessboard to be in "equilibrium". We define *equilibrium* to mean that no piece can capture another piece given an unbounded number of turns. Given a required set of pieces, the pieces can be placed anywhere on the board. All regular chess rules hold *except* for the rules governing the special powers of the king. The specific set of rules for our game are as follows; the first four rules are regular chess rules and the last is a rule specific to our game:

1. All pieces move according to regular chess rules. (For example, black pawn moves down and captures diagonally down; white pawn moves up and captures diagonally up.)
2. No special rules. No pawn promotion, castling, en passant, etc.
3. A player *must* move if they are able to move.
4. White goes first. Players take turns moving one piece of their color each turn.
5. The king has no special status. Check and checkmate have no meaning.

The goal of the black player is to win by taking any white piece and vice versa for the white player. If neither player can win, the next best result is a tie where neither player can capture any of the other player's pieces. We assume optimal gameplay by the two players, so for example they never move a piece to a position that can be immediately captured. The puzzle solver is given a starting board and a set of required pieces to place on the board. The puzzle solver must then place the required pieces in an equilibrium configuration on the board. An arrangement of pieces that leads to a tie even when players are playing optimally with unbounded number of moves is what we seek in Chess Equilibrium.

Example Game Play To help you become familiar with our puzzles, we provide some examples of Chess Equilibrium puzzles and prove their solutions generalize to

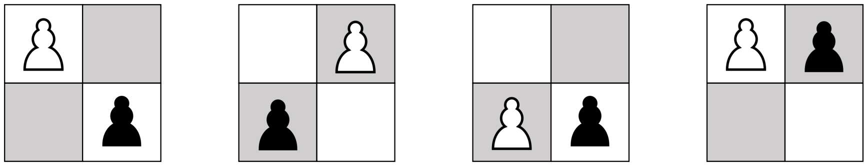


Figure 1 Two pawns of opposite colors are in equilibrium as long as both pawns start in the same row (so can only pass each other) or the white pawn is on the top row (so cannot move or capture/be captured) or the black pawn is on the bottom row (so cannot move or capture/be captured).

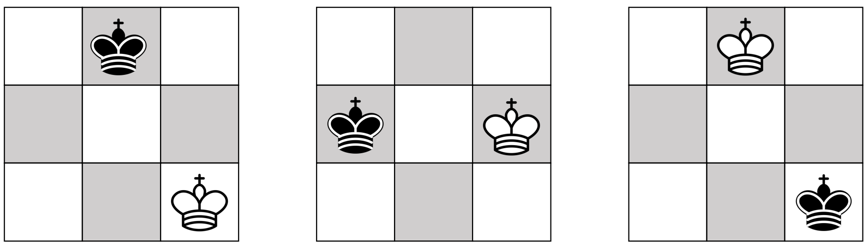


Figure 2 For two kings of opposite color, any starting placement where the kings are not on adjacent squares is a valid equilibrium configuration. We prove this statement in Theorem 1. The figure shows some example solutions to the two-king 3×3 Chess Equilibrium puzzle.

larger classes of puzzles. The following are two simple, solved puzzles which generalize to larger boards and numbers of pieces. The first, Figure 1, asks for the placement of one black pawn and one white pawn on a 2×2 board. The second, Figure 2, asks for the placement of one black king and one white king on a 3×3 board.

The case of two kings is special in that any placement such that the two kings are not in neighboring squares (where we consider diagonally adjacent kings to also be neighbors) is an equilibrium placement on any $n \times n$ board. For $n < 3$, there is no such placement. In the simplest case of a 3×3 board, any position in which the kings are not adjacent also has neither king on the center square, and is an equilibrium arrangement because the two kings can just move endlessly around the edge of the board. Theorem 1 proves this for any size of board.

Theorem 1. *For any $n \times n$ board, any placement of one white and one black king such that the two kings are not in neighboring squares is an equilibrium solution.*

Proof. Define a king's *territory* to be the squares that the king could capture on their next turn. At any point in the game, a king may be in one of three possible positions: in a corner, against a wall (but not in a corner), and away from a wall. In each case, the king's territory consists of 3, 5, and 8 squares, respectively. Also, for each case, the maximum possible number of overlapping squares between the current king's territory and a non-neighboring other king's territory is 2, 3, and 3, respectively. Therefore, no matter where a king is placed (as long as it is not neighboring the other king), it must have at least 1 square where it may move to without being captured on the next turn. Because there is an infinite move sequence and no initial possible capture, no capture is ever possible under optimal play because all pieces have the same moves, so capturing would require moving the attacking piece into a capturable position. □

We can further generalize this theorem to a solution for $2k$ kings, half of each color, for any $n \times n$ board where $n \geq \frac{1}{2} \cdot \left(\sqrt{8k+9}+1\right)$. Theorem 2 proves that

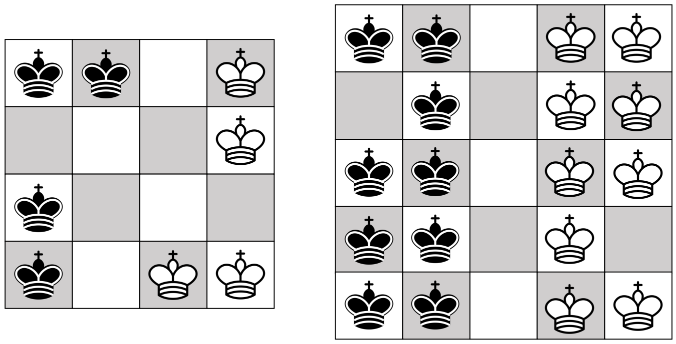


Figure 3 An example of a valid equilibrium placement of $2k$ kings where $k = 4$ (left) and where $k = 9$ (right). The right example can also be extended to $k < 9$ by removing some of the kings.

there always exists an obvious equilibrium placement for such puzzles.

Theorem 2. *For any $k \geq 1$, k black kings and k white kings have an equilibrium placement on an $n \times n$ board where $n \geq \frac{1}{2} \cdot \left(\sqrt{8k+9}+1\right)$ and n is odd.*

Proof. First, the board must be large enough to satisfy the conditions presented in this proof. The simple placement that always results in an equilibrium is to place one king of each color in a corner of the square board and place another king of the same respective color next to the one king such that it is one blank square away from the corner king and is against the wall; see Figure 3. The other kings may be placed anywhere except the blank square, so long as no opposite color kings are neighbors. This is an equilibrium configuration because both players may simply move their corner kings into and out of the blank square, and as in Theorem 1, no captures become possible because the pieces all have the same moves.

The sufficient condition for n can be proved by placing all kings of one color on one side of the board and the other kings on the other side of the board; Figure 3 exhibits one such placement. Specifically, to get our lower bound on n using the above reasoning, all kings of one color can be placed adjacent to one another on one half of the board and the kings of the other color can be placed on the other half of the board. Then, we create one blank square on each half for one king of each color to move back and forth. Finally, we have one middle column of unoccupied squares in between the two halves. To achieve this configuration, we need n to be odd and large enough so that $n^2 \geq 2k + 2 + n$ where $2k$ is the total number of kings, 2 is for the additional blank square for each player, and n is the middle column of unoccupied squares. Solving $n^2 - n - 2k - 2 \geq 0$ gives $n \geq \frac{1}{2} \cdot \left(\sqrt{8k+9}+1\right)$. □

Theorem 2 may be generalized to show that an equilibrium placement exists for cases when the two players have unequal number of kings, but each player has at least two kings, on a large enough board. We will not go into details for this case as it uses the same method of placement but with a slightly different board-size bound.

Example Chess Equilibrium Puzzle and Analysis

As preparation for solving Chess Equilibrium puzzles, we first include a full puzzle, a possible solution, and a detailed analysis of the configuration of a correct placement of pieces. Figure 4 gives the puzzle, and Figure 5 gives a solution.

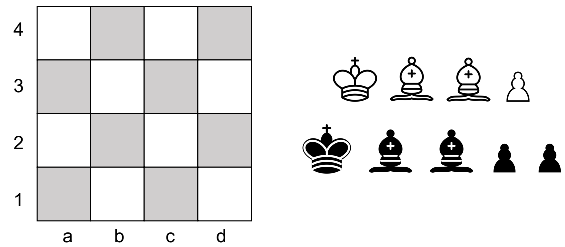


Figure 4 Place the required pieces on the board to reach a Chess Equilibrium.

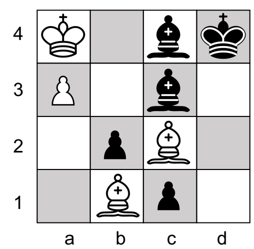
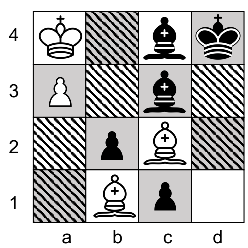
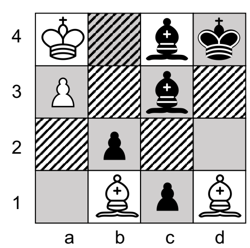


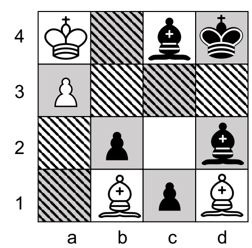
Figure 5 A solution to the puzzle in Figure 4.



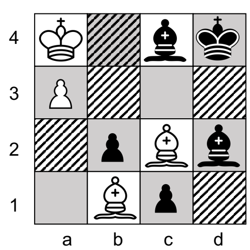
(a) White to move.



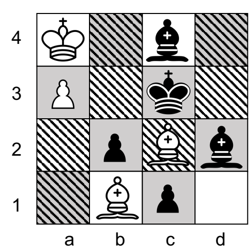
(b) Black to move.



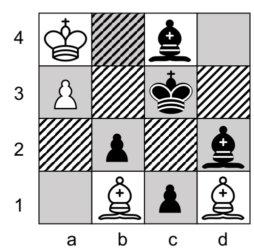
(c) White to move.



(d) Black to move.



(e) White to move.



(f) Black to move.

Figure 6 Diagonal lines indicate squares that the opponents' pieces can capture, so are not safe for the current player to move to.

Solution Explanation Because white moves first, it must first find a valid move. The only valid move is Bd1; see Figure 6a which shows all the squares that black can capture on the next turn. Then, on the next turn, the only valid move for black is Bd2; see Figure 6b which shows all the squares that white can capture on the next turn. After this move, there are two valid moves for white: either bishop can move to c2. However, if the white bishop in b1 moves to c2, then the black bishop in c4 can move to a2 on the next turn, which forces a white piece to be captured on the following turn (refer to Figure 6c). Thus, the bishop in b1 cannot move to c2, so the bishop in d1 must move to c2. Afterward, either the king in d4 or the bishop in d2 can move to c3 (Figure 6d). If the bishop moves to c3, then we reach the original configuration of Figure 5. Otherwise, on the next turn, the only valid move is again Bd1 (Figure 6e). Finally, the king in c3 must return to its original location (Figure 6f), reaching the configuration in Figure 6c again.

Chess Equilibrium Puzzles

Here, we provide some larger (and less trivial) puzzles for your enjoyment. We welcome new and creative solutions beyond the solutions we provide at the end of this paper. We also welcome new Chess Equilibrium puzzles!

Problem 1: Shown in Figure 7.

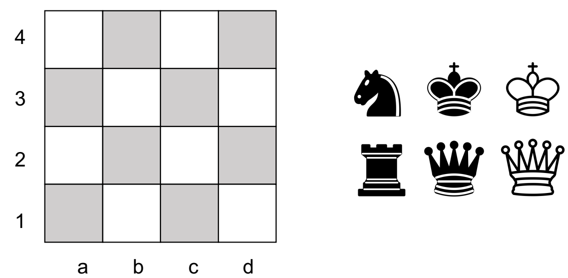


Figure 7 Place the required pieces on the board to reach a Chess Equilibrium.

Problem 2: Shown in Figure 8.

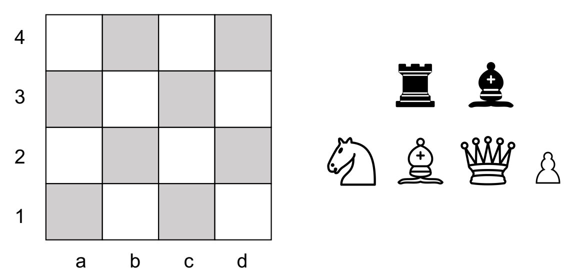


Figure 8 Place the required pieces on the board to reach a Chess Equilibrium.

Other Chess Puzzles

Chess puzzles intrigue chess lovers, puzzle lovers, and mathematicians alike. Some of the more famous examples of chess puzzles include the retrograde and mate-in-k chess puzzles [**7, 8**]. In retrograde chess, puzzle solvers are given a chessboard configuration in the middle of a game and asked to answer some question about the history of the game from the initial setup that resulted in the current configuration of the board [**4**]. Such questions could include "determine a legal sequence of four moves that lead to the current placement"; "is the queen on b1 promoted?"; or "there exists exactly one square where the king can be; which one is it?". Mate-in-k puzzles, on the other hand, challenge the puzzle solver to find the set moves that forces mate in k moves on a given board.

Both forms of puzzles can be generalized to $n \times n$ boards (for a positive integer n) and larger numbers of pieces (more than the 16 pieces of each color). The *puzzle size* is defined to be the number n^2 of squares of the board, which is also an upper bound on the total number of pieces. Theoretical computer scientists have long wondered how complex generalized versions of these puzzles are. The complexity of these puzzles is defined formally as how computationally hard it is to solve the puzzle. Puzzles are *computationally hard* if one can reduce a known computationally hard problem to versions of the puzzle on $n \times n$ boards such that solving the puzzle solves the original hard problem. Such a *reduction* shows that solving the puzzle cannot be easier than solving the original problem. Hard problems that we usually use for these reductions are generally NP-hard or PSPACE-hard, which suggest that no algorithm can solve them in an amount of time polynomial in the input size. For more details regarding these complexity classes, refer to Arora and Barak [**1**].

Bodlaender [**2**] proved that generalized retrograde chess puzzles are NP-hard. This result left open whether these puzzles are in NP. As a step toward answering this question in the negative, Shitov [**6**] created a generalized retrograde chess puzzle whose solution requires an exponential sequence of moves. In other words, given any puzzle size n, Shitov [**6**] can create two positions A and B such that going from position A to position B requires a number of moves that is exponential in n. Recently, Brunner et al. [**3**] settled the problem by proving that generalized retrograde chess is PSPACE-complete. Mate-in-k puzzles, on the other hand, can be solved in time polynomial in n when $k = O(1)$, but become PSPACE-complete when k is polynomial in the board size n as shown by Storer [**9**].

The chess puzzles that are perhaps most similar to Chess Equilibrium are the Eight Queens Problem or the Nine Queens Problem (and its variants). These types of puzzles are collectively called Chess Non-attacking Puzzles; for examples of such puzzles, see Gardner [**4**] and Levitin and Levitin [**5**]. These puzzles ask to find the maximum number of pieces that can be placed on a board with it being possible for any piece to immediately attack another piece, which is a necessary condition for equilibrium. But the solutions to these puzzles are not necessarily solutions to the corresponding Chess Equilibrium puzzles, because of the rule that a player must move if they are able and the rule that players must take turns moving their pieces. (Not to mention that the original Eight Queens and Nine Queens problems did not consider pieces of different colors.) As we see in the solutions to the problems, the chess rule that white goes first is important in establishing equilibrium of certain Chess Equilibrium configurations. Furthermore, the rule that a player must move if they can move serves to establish a unique and interesting type of equilibrium in this class of puzzles.

Solutions

Problem 1 Shown in Figure 9. The key here is that white moves first. Because the queen in d1 cannot eliminate her threat (the knight in c3) and also cannot capture a piece, the only thing she can do is move out of the way. The only available move is Qd3. On the next turn, the knight must move to a4 to avoid being captured by the

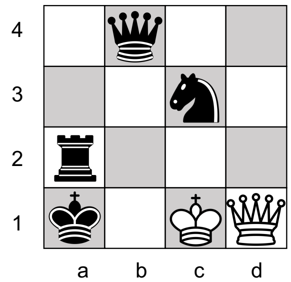


Figure 9 A solution to Problem 1.

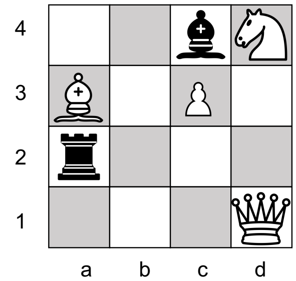


Figure 10 A solution to Problem 2.

queen. Then, the only available move is Qd1. The knight then has to move to its starting square (c3) to avoid being captured by the queen.

There's some explanation needed to justify why the white king does not move into d1 on white's second move. The reason is that this move is not optimal because the knight could then move so that, two moves afterward, either the queen or the king will be captured. Specifically, the set of moves is as follows: after Qd3, (1) Na4, (2) Kd1, (3) Nb2. Nb2 results in a fork between Qd3 and Kd1.

Problem 2 Shown in Figure 10. The key for this solution is that the first move must be Bb4. If instead the first move is Bc1, then the next optimal move would be Ra1 which results in a loss for white because either the bishop in c1 or the queen in d1 will be captured. After Bb4, the only available move is Rb2. After Rb2, the bishop moves back to a3. Then, the only available move is Ra2 which returns to the starting configuration in Figure 10.

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Summary What happens when the *only* goal in a chess game is to capture at least one piece of the opposite side? Can both sides live peacefully in an equilibrium where neither can capture the other's pieces? In this short paper, we develop a new set of puzzles which we call *chess equilibrium* puzzles on this premise. We explain the rules of the game, analyze puzzles that have obvious and generalizable solutions, and provide several interesting puzzles for the reader to solve (solutions are provided at the end). Our puzzles provide an exciting twist to the realm of traditional chess puzzles.

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