Degenerative Coordinates in 22.5° Grid System

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Abstract

We consider the construction of points within a square of paper by drawing a line (crease) through an existing point with angle equal to an integer multiple of 22.5°, which is a very restricted form of the Huzita–Justin origami construction axioms. We show that a point can be constructed by a sequence of such operations if and only if its coordinates are both of the form $(m+n\sqrt{2})/2^{\ell}$ for integers m, n, and $\ell \geq 0$, and that all such points can be constructed efficiently. This theorem explains how the restriction of angles to integer multiples of 22.5° forces point coordinates to degenerate into a reasonably controlled grid.

1 Introduction

The crease patterns for many origami models are designed within an angular grid system of $90°/n$, for a nonnegative integer n. Precisely, in this system, every (pre)crease passes through an existing reference point in the direction of $m(90/n)°$ for some integer m, and every reference point is either $(0,0)$, $(1,0)$, or an intersection of already constructed (pre)creases. For example, 45° ($n=2$), 30° ($n=3$), 22.5° ($n=4$), 18° ($n=5$), and 15° ($n=6$) grid systems are known to be useful for the design of origami.

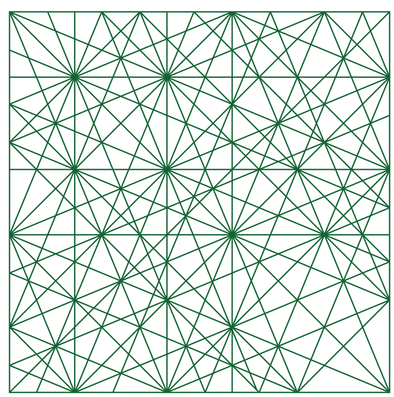

Figure 1: Maekawa-gami: 22.5° grid.

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In particular, the 22.5° grid system has been used for centuries—one of the oldest example is the classic origami crane—and the system keeps producing complex but organized origami expressions such as the *Devil* (1980) by Jun Maekawa [Maekawa and Kasahara 83, Maekawa 07, pp. 146–154] and the *Wolf* (2006) by Hideo Komatsu [Komatsu 06]. Toshikazu Kawasaki calls this system "Maekawa-gami". Figure 1 shows a square filled with several precreases in the 22.5° grid system.

Why are these angular grid systems so useful? A striking feature of Figure 1 is that there are many ways to construct the same point, and as a consequence, many alignments among points and lines. Intuitively, this *degeneracy* of the construction system helps tame the complexity of crease patterns designed within the system.

In this paper, we formalize this notion of degeneracy and organized complexity by characterizing the coordinates of reference points in the 22.5° grid system of the unit square as those points (x, y) with $x, y \in \mathcal{D}_{\sqrt{2}}$, where

$$\mathcal{D}_{\sqrt{2}} = \left\{ \frac{m + n\sqrt{2}}{2^\ell} \;\middle|\; \text{integers } m, n, \text{and } \ell \geq 0 \right\}.$$

In particular, we establish degeneracy by proving that all constructible points fall into $\mathcal{D}^2_{\sqrt{2}} = \mathcal{D}_{\sqrt{2}} \times \mathcal{D}_{\sqrt{2}}$, and establish universality by proving that all points in $\mathcal{D}^2_{\sqrt{2}}$ can be constructed. In the latter result, the number of required operations is linear in the bit complexities of x and y, where the *bit complexity* of a number $\frac{m+n\sqrt{2}}{2^\ell} \in \mathcal{D}_{\sqrt{2}}$ is $\lg(2+|m|)+\lg(2+|n|)+\ell$.

1.1 Model

More precisely, we consider the following models of 22.5° grid construction.

The initial set of points can be either two marks on the x axis, $\{(0,0), (1,0)\}$, or all four corners of the square, $\{(0,0),(1,0),(0,1),(1,1)\}$. The choice between these two options does not affect the results; for the strongest results, we use the former set for our construction, and the latter set for proving degeneracy.

The *grid-line construction* is to draw a line through an existing point, at an angle of $k\,22.5°$ with respect to the x axis, for an integer $k \in \{0, 1, \ldots, 7\}$. This line also defines newly constructed points by its intersections with all other drawn lines.

A grid-line construction can be simulated by $O(1)$ applications of Huzita–Justin axioms. Recall that the Huzita–Justin axioms [Huzita and Scimemi 89, Justin 89, Demaine and O'Rourke 07, chap. 19] include the ability to fold the line through two given points, fold the perpendicular bisector of two points, fold the angular bisector of two given lines, fold the perpendicular to a given line passing through a given point, and two operations constructing tangents to parabolas. In fact, we need only two of these axioms:

folding the angular bisector of two lines, and folding through a point and perpendicular to a line. Then we can perform a grid-line construction by constructing two lines through the point perpendicular to the two axes, then bisecting one of the 90° angles once or twice to obtain the desired integer multiple of 22.5°.

To avoid this constant-factor overhead, our construction will simultaneously adhere to the constraints of both the 22.5° grid system and these two Huzita–Justin axioms. Thus, every operation we perform in the construction will bisect an angle at an existing point, or be perpendicular to an existing line and through an existing point, and furthermore the constructed line's angle with respect to the x axis will be an integer multiple of 22.5°. Note that these two operation types are indeed special cases of grid-line constructions, in addition to corresponding to real origami constructions. We call these operations *grid-line axioms*.

2 Construction

In this section, we give a universal construction algorithm for points in $\mathcal{D}^2_{\sqrt{2}}$:

Theorem 1 *We can construct any point in $\mathcal{D}^2_{\sqrt{2}}$ by a sequence of grid-line axioms whose length is linear in the bit complexities of the two coordinates.*

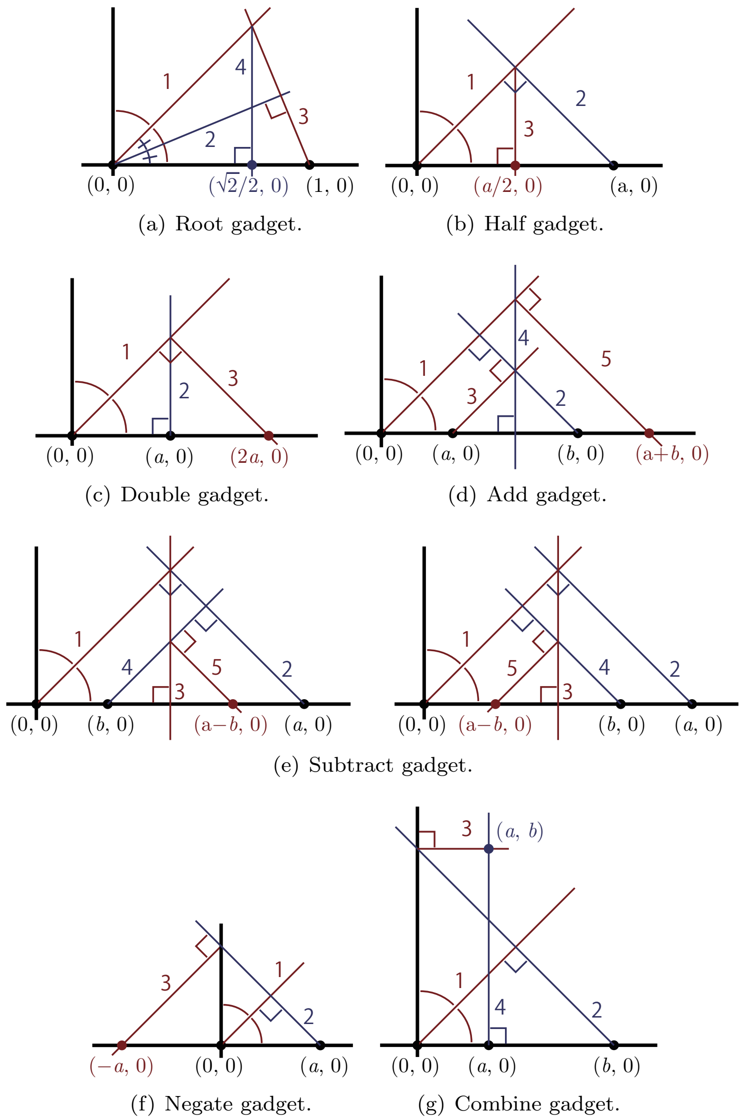
The construction constructs each coordinate of the target point separately. Thus we focus mostly on the construction of a single number in $\mathcal{D}_{\sqrt{2}}$, as measured by the distance from the origin of a point along the x axis. To perform such a construction, we combine several gadgets for constructing individual numbers and performing arithmetic on numbers:

1. Root gadget: Construct the number $\frac{\sqrt{2}}{2}$. The gadget essentially reflects the diagonal down to the axis (Figure 2(a)).

2. Half gadget: Given a positive number a, construct $a/2$. The gadget uses 45° lines to construct the midpoint (Figure 2(b)). Note that, given all of the Huzita–Justin axioms, we could instead simply use a perpendicular bisector to construct the desired point.

3. Double gadget: Given a positive number a, construct $2\,a$. The gadget essentially reflects a copy of a (Figure 2(c)).

4. Add gadget: Given two positive numbers a and b, construct $a + b$. The gadget essentially reflects a copy of the smaller integer to the right of the larger integer (Figure 2(d)).



(a) Root gadget. (b) Half gadget.

(c) Double gadget. (d) Add gadget.

(e) Subtract gadget.

(f) Negate gadget. (g) Combine gadget.

Figure 2: Gadgets. Before applying any gadgets, we construct the x axis from the given points $(0, 0)$ and $(1, 0)$, and construct the y axis as perpendicular to the x axis through $(0, 0)$.

5. Subtract gadget: Given two positive numbers a and b with $a > b$, construct $a - b$. The gadget computes $a/2$ and then essentially reflects b

around $a/2$, which gives $2(a/2) - b = a - b$. (Figure 2(e)). In fact, the construction has two cases, depending on whether $b \leq a/2$ or $b \geq a/2$

6. Negate gadget: Given a positive number a, construct $-a$. The gadget essentially reflects a to the left of the y axis (Figure 2(f)).

Lemma 2 *We can construct any number in $\mathcal{D}_{\sqrt{2}}$ by a sequence of grid-line axioms whose length is linear in the bit complexity of the number.*

Proof: Consider a number $x = (m + n\sqrt{2})/2^\ell \in \mathcal{D}_{\sqrt{2}}$.

We construct $|m|$ using a standard repeated doubling trick (analogous to repeated squaring [Cormen et al. 09, Sect. 31.6]). If $|m|$ is even, recursively construct $|m|/2$ and then use the double gadget. If $|m|$ is odd, recursively construct $(|m| - 1)/2$, then use the double gadget to obtain $|m| - 1$, and then use the add gadget to add 1. In the base cases, we already have the constant 0 and 1. The number of operations is $O(\lg(2 + |m|))$.

Similarly, we can construct $|n|\sqrt{2}$ by using $\sqrt{2}$ instead of 1 as a base case. We can construct $\sqrt{2}$ via the root gadget followed by the double gadget. The number of operations is $O(\lg(2 + |n|))$.

Finally, we combine the two values $|m|$ and $|n|\sqrt{2}$ using an add or subject gadget to obtain $|m + n\sqrt{2}|$, then use the half gadget ℓ times to construct $|x|$, and then use the negate gadget if x is negative. (We can negate only at the end because the other gadgets are designed for positive numbers.) □

Proof of Theorem 1: Consider a point $p = (x, y) \in \mathcal{D}^2_{\sqrt{2}}$, where $x = (m_x + n_x\sqrt{2})/2^{\ell_x})$ and $y = (m_y + n_y\sqrt{2})/2^{\ell_y})$. We use Lemma 2 to construct the numbers x and y along the x axis. Then we copy the value y onto the y axis using a 45° line, and then find the intersection of two perpendiculars to find the point p, as shown in Figure 2(g). □

For points in the unit square, we can restrict to working within the square of paper:

Lemma 3 *We can construct any number in $[0, 1] \cap \mathcal{D}_{\sqrt{2}}$ by a sequence of grid-line axioms, with all intermediate points in $[0, 1]$, whose length is linear in the bit complexity of the number.*

Proof: Consider a number $x = (m + n\sqrt{2})/2^\ell \in \mathcal{D}_{\sqrt{2}}$ with $0 \leq x \leq 1$. Let $s = 1 + \max\{\lg |m|, \lg(|n|\sqrt{2})\}$ be the smallest integer such that $|m|/2^s \leq \frac{1}{2}$ and $(|n|\sqrt{2})/2^s \leq \frac{1}{2}$. First we construct $S = 1/2^s$ by s half gadgets. Then we apply the construction in Lemma 2 but using S in place of 1. Because all numbers in this construction are at most $|m| + |n|\sqrt{2}$, but everything is scaled by $1/S$, all intermediate values are in $[0, 1]$. Thus we obtain

$(m + n\sqrt{2})/2^{\ell+s}$. Finally we scale back up using s double gadgets. (For practicality, we could have saved $\min\{\ell, s\}$ half/double gadget pairs, but this only affects the constant factor.) The total number of operations is $O(\lg(2+|m|) + \lg(2+|n|) + \ell)$ because $s = O(\lg(2+|m|) + \lg(2+|n|))$. □

Theorem 4 *We can construct any point in $[0,1]^2 \cap \mathcal{D}^2_{\sqrt{2}}$ by a sequence of grid-line axioms, with all intermediate points in $[0,1]^2$, whose length is linear in the bit complexities of the two coordinates.*

Proof: Simply follow the proof of Theorem 1 but using Lemma 3 in place of Lemma 2. □

3 Degeneracy

In this section, we show the degeneracy in the grid system: all constructible points are restricted within $\mathcal{D}^2_{\sqrt{2}}$.

Theorem 5 *Every point constructible by a sequence of grid-line constructions, starting from the corners of a unit square, is in $\mathcal{D}^2_{\sqrt{2}}$.*

The proof is by induction. In the base case, we start from the four points at the corners of the square, $\{(0,0),(1,0),(0,1),(1,1)\}$, which are in $\mathcal{D}^2_{\sqrt{2}}$. For the induction step, we prove that any newly constructed points by grid-line constructions from points in $\mathcal{D}^2_{\sqrt{2}}$ are also in $\mathcal{D}^2_{\sqrt{2}}$. In order to cull duplicate combinations, we first extend the system to allow 45° rotation.

Lemma 6 *Points in $\mathcal{D}^2_{\sqrt{2}}$ are closed under $k\,45°$ rotation about the origin, for any $k \in \{0,1,\ldots,7\}$.*

Proof: For a point $(x,y) \in \mathcal{D}^2_{\sqrt{2}}$, its 45° rotation (x', y') is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Because $\mathcal{D}_{\sqrt{2}}$ is closed under addition, subtraction, and multiplication, x' and y' are in $\mathcal{D}_{\sqrt{2}}$. By induction, a point produced by $k\,45°$ rotation is in $\mathcal{D}^2_{\sqrt{2}}$. □

Lemma 7 *Two lines made by grid-line constructions from points in $\mathcal{D}^2_{\sqrt{2}}$ have their intersection point in $\mathcal{D}^2_{\sqrt{2}}$.*

Proof: Consider two grid-line constructions from points in $\mathcal{D}_{\sqrt{2}}^2$, say, line L_0 from point (x_0, y_0) and line L_1 from point (x_1, y_1). For $i \in \{0,1\}$, we can define line L_i as points (x, y) satisfying $s_i(x - x_i) + t_i(y - y_i) = 0$, where (s_i, t_i) is a vector perpendicular to the line and thus t_i/s_i is the slope. For the lines to have an intersection, they must not be parallel, i.e., $s_0 t_1 - s_1 t_0 \neq 0$. The intersection point (x, y) is given by

$$\begin{cases} s_0(x - x_0) + t_0(y - y_0) = 0 \\ s_1(x - x_1) + t_1(y - y_1) = 0, \end{cases} \tag{1}$$

which can be represented in matrix form as

$$\begin{bmatrix} s_0 & t_0 \\ s_1 & t_1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_0 x_0 + t_0 y_0 \\ s_1 x_1 + t_1 y_1 \end{bmatrix}.$$

By Cramer's Rule, the solution is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{s_0 t_1 - t_0 s_1} \begin{bmatrix} t_1 & -t_0 \\ -s_1 & s_0 \end{bmatrix} \begin{bmatrix} s_0 x_0 + t_0 y_0 \\ s_1 x_1 + t_1 y_1 \end{bmatrix}. \tag{2}$$

Because $\mathcal{D}_{\sqrt{2}}$ is closed under multiplication and addition, it suffices to show that $1/(s_0 t_1 - t_0 s_1)$ is in $\mathcal{D}_{\sqrt{2}}$ for any combination of vectors (s_0, t_0) and (s_1, t_1).

There are eight possible orientations for each vector (s_i, t_i) in the 22.5° system, given by representative vectors

$$\begin{aligned} (s,t) \in \; & \{(1,0), (1, -1+\sqrt{2}), (1,1), (-1+\sqrt{2}, 1), \\ & (0,1), (1-\sqrt{2}, 1), (-1,1), (-1, -1+\sqrt{2})\}. \end{aligned} \tag{3}$$

Note that we do not need to list the negations of these vectors, as it suffices to capture all line slopes, not signed directions, for the line equations in (1).

Instead of checking every pair of slopes ($\binom{8}{2} = 28$ patterns), we can reduce the possible combinations down to 10 cases by using Lemma 6 to perform $k\,45°$ rotation in advance. Namely, if one of the lines has angle $k\,45°$ for an integer k, then we rotate by $-k\,45°$ to give that line orientation $(1,0)$, and obtain seven possible cases for the other line (Figure 3, left); and if both lines have angles that are not integer multiples of 45°, then we rotate so that one of them is 22.5° and obtain three cases for the other line (Figure 3, right).

Table 1 computes $1/(s_0 t_1 - t_0 s_1)$ for each of these ten cases. In all cases, the value is in $\mathcal{D}_{\sqrt{2}}$, so $(x, y) \in \mathcal{D}_{\sqrt{2}}^2$. □

Proof of Theorem 5: Consider a sequence of grid-line constructions $\ell_1, \ell_2, \ldots, \ell_n$. By definition, each ℓ_k is a line through an existing point,

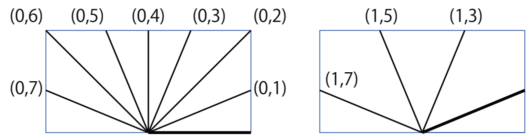


Figure 3: Possible combinations of directions after suitable $k\,45°$ rotation. Labels refer to indices into the list (3) of possible (s,t) vectors, starting at index 0.

case	(s_0, t_0)	(s_1, t_1)	$1/(s_0 t_1 - t_0 s_1)$
(0, 1)	$(1, 0)$	$(1, -1 + \sqrt{2})$	$1 + \sqrt{2}$
(0, 2)	$(1, 0)$	$(1, 1)$	1
(0, 3)	$(1, 0)$	$(-1 + \sqrt{2}, 1)$	1
(0, 4)	$(1, 0)$	$(0, 1)$	1
(0, 5)	$(1, 0)$	$(1 - \sqrt{2}, 1)$	1
(0, 6)	$(1, 0)$	$(-1, 1)$	1
(0, 7)	$(1, 0)$	$(-1, -1 + \sqrt{2})$	$1 + \sqrt{2}$
(1, 3)	$(1, -1 + \sqrt{2})$	$(-1 + \sqrt{2}, 1)$	$(1 + \sqrt{2})/2$
(1, 5)	$(1, -1 + \sqrt{2})$	$(1 - \sqrt{2}, 1)$	$(2 + \sqrt{2})/2^2$
(1, 7)	$(1, -1 + \sqrt{2})$	$(-1, -1 + \sqrt{2})$	$(1 + \sqrt{2})/2$

Table 1: Case analysis of slopes. The leftmost column refers to labels in Figure 3.

either an original corner of the square $\{(0,0),(1,0),(0,1),(1,1)\}$, or defined by an intersection between two lines ℓ_i and ℓ_j for $i, j < k$.

We claim by induction that each line ℓ_k is a grid-line construction from a point in $\mathcal{D}^2_{\sqrt{2}}$. If ℓ_k is constructed from a corner of the unit square, this claim follows because $(0,0),(1,0),(0,1),(1,1) \in \mathcal{D}^2_{\sqrt{2}}$. Otherwise, ℓ_k is constructed from the intersection of two lines ℓ_i and ℓ_j with $i, j < k$. By induction on k, both ℓ_i and ℓ_j are grid-line constructions from points in $\mathcal{D}^2_{\sqrt{2}}$. Thus Lemma 7 applies, and the intersection point defining ℓ_k is in $\mathcal{D}^2_{\sqrt{2}}$.

Finally, the points formed by the sequence of grid-line constructions $\ell_1, \ell_2, \ldots, \ell_n$ are the intersections between two lines ℓ_i and ℓ_j. By the claim above, Lemma 7 applies to show that every such point is in $\mathcal{D}^2_{\sqrt{2}}$. $\square$

4 Conclusion

We have characterized the degeneracy of points constructible in the 22.5° grid system. The restricted form we establish for constructible coordinates indicates that there are many possible ways to construct a point, which should tend to lead to fortuitous alignments of creases. For example, these alignments make it easier to choose creases so that they meet at vertices of degree at least 4, as necessary for flat foldability, whereas without a grid system, generically chosen lines would meet only in pairs and would not satisfy Kawasaki's condition for local flat foldability. Furthermore, our algorithms show that any desired point in the grid system can be constructed efficiently using just two types of origami operations, in a sequence of length linear in the bit complexity of the coordinates. These results provide mathematical support for why practical origami design uses grid systems.

A simple extension of our theory is to the situation of 22.5° grid-line constructions starting from a length ratio of $(m + n\sqrt{2})/(m' + n'\sqrt{2})$, for some integers m, n, m', n'. For example, Maekawa's wani (alligator/crocodile) [Maekawa and Kasahara 83] starts by dividing the paper side in thirds, and then works on the 22.5° grid. This situation commonly results from the origami design process (e.g., grafting or adjusting flap lengths), because it effectively adjusts the size of the square of paper. Our theory can capture these situations simply by viewing the square as having side length $m' + n'\sqrt{2}$ instead of 1, that is, by scaling all coordinates by a factor of $m' + n'\sqrt{2}$, thereby placing the constructed points back on the 22.5° grid.

An obvious direction for future research is characterizing $90°/n$ grid systems for $n \neq 4$. In particular, modern origami design has explored the 15° grid system lately. For results in this direction, see the follow-up paper [Butler et al. 10].

References

[Butler et al. 10] Steve Butler, Erik D. Demaine, Ron Graham, and Tomohiro Tachi. "Constructing points through folding and intersection." Manuscript, 2010.

[Cormen et al. 09] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms*, Third edition. MIT Press, 2009.

[Demaine and O'Rourke 07] Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra.* Cambridge University Press, 2007.

[Huzita and Scimemi 89] Humiaki Huzita and Benedetto Scimemi. "The Algebra of Paper Folding (Origami)." In *Proceedings of the 1st International Meeting of Origami Science and Technology*, edited by H. Huzita, pp. 215–222. Ferrara, Italy, 1989.

[Justin 89] Jacques Justin. "Resolution par le Plaige de L'equation du Troisieme Degre et Applications Geometriques." In *Proceedings of the 1st International Meeting of Origami Science and Technology*, edited by H. Huzita, pp. 251–261. Ferrara, Italy, 1989.

[Komatsu 06] Hideo Komatsu. "Wolf." *Origami Tanteidan Magazine* 17:3 (2006), 22–32.

[Maekawa and Kasahara 83] Jun Maekawa and K. Kasahara. *Viva Origami! (in Japanese)*. Sanrio Publications, Tokyo, 1983.

[Maekawa 07] Jun Maekawa. *Genuine Origami.* Nichibo Shuppansha, 2007. English version published in 2008.