18.435/6.848 Theory of Parallel and VLSI Computation	Fall 1997

Problem Set #4

Due: Wednesday, October 29 — *Prof. Leighton*

Reading (for lectures 11-13): Sections 3.1 – 3.3, 3.7–3.8 of text, but skip starred sections.

Problems (for lectures 11–13):

1. Give an explicit embedding of the $4 \times 8 \times 16$ array in the 512-node hypercube. Be sure to specify where node (i, j, k) of the array gets mapped to in the hypercube.

*2. Show how to find $\lfloor \frac{\log N}{2} \rfloor$ edge-disjoint Hamiltonian cycles in an N node hypercube.

*3. Analyze the performance of the following randomized algorithm for embedding a tree into an N-node hypercube. The root of the tree is embedded into node $0 \ldots 0$ of the hypercube. If a tree node x is embedded into hypercube node v, and x spawns a child y, then y is embedded into a randomly chosen neighbor of v in the hypercube. Show that if we apply this algorithm to an $(N-1)$-node complete binary tree, then we can expect $\Omega(N^{\epsilon})$ leaves of the tree to be mapped to hypercube node $0 \ldots 0$, where $\epsilon > \log(2\sqrt{2}/e)$. (Hint: Show that any particular leaf is mapped to $0 \ldots 0$ with probability $p \geq \Omega(N^{\epsilon-1})$, where p is the probability that each of $\log N$ boxes will contain an even number of balls after $\log N - 1$ balls are thrown at random into the boxes.) You may assume that $\log N$ is odd. What happens when $\log N$ is even? (3.35)

*4. Show how to embed the $N \times N$ pyramid in the $2N^2$ node hypercube with constant dilation and constant congestion. (This was once a real $$ problem.) (3.68)

5. Show that r perfect outshuffles return an N-card deck (N is assumed to be even) to its original order if and only if $2^r \equiv 1 \bmod (N-1)$. Now, prove that eight perfect outshuffles are required and sufficient to restore a deck of 52 cards to its original order. How many outshuffles are needed to restore a 50-card deck to its original order? (3.153,3.154,3.155)

6. Construct a 5-bit Gray code and a 32-bit de Bruijn sequence.

7. Show that an N-node de Bruijn graph contains an $(N-1)$-node complete binary tree as a subgraph. In addition, show that an N-node de Bruijn graph contains a $(2N-1)$-node complete binary tree where every node is used precisely once as a leaf, and at most once as an internal node of the tree. (3.172, 3.173)

8. Describe the network that you get if you remove the nodes on level i of an r-dimensional butterfly.

9. Given a $\sqrt{N} \times \sqrt{N}$ matrix of inputs $X = (x_{ij})$, a two-dimensional FFT on X consists of computing an FFT on the $\sqrt{N}$ values in each column of X, and then computing an FFT on the $\sqrt{N}$ values in each row of the resulting matrix. Show how to compute a two-dimensional FFT in $\log N$ steps with one pass through a $\log N$-dimensional butterfly when N is a power of 4. How fast can a two-dimensional FFT be computed on a N-node hypercube or the shuffle exchange graph? (3.344)

10. The *odd graph* has been proposed as a candidate for the network for the Tera computer. Each node corresponds to a binary number with $2j + 1$ digits of which j are 0 and $j + 1$ are 1. Two nodes are connected if one number may be obtained from the other by complementing all of the bits except a single 1 which is left a 1. How many nodes would this network have? (Hint: Use Stirling's formula.) What is the degree of each node? What is the diameter of the graph? *Extra Credit: What is the bisection width?*

Problems For Extra Credit

\$ 11. Is the *odd graph* computationally equivalent (up to constant slowdown) to a hypercube?

\$\$ 12. Can a shuffle-exchange graph be embedded one-to-one in a hypercube with constant expansion and constant dilation? (3.73)

\$\$ 13. Is it possible to embed every N-node binary tree in an N-node hypercube with dilation 1 and load $o(\log N)$? Is there any binary tree that requires load $\Omega(1)$? Does the M-node hypercube contain every N-node binary tree as a subgraph for $M = o(N \log N)$? For $M = O(N)$? (3.41)

\$\$\$ 14. Can every N-node binary tree be embedded one-to-one in an N-node hypercube with dilation at most 2? (3.76)

\$\$ 15. Can every N-node k-dimensional array be embedded $1-1$ in a $2^{\lceil \log N \rceil}$-node hypercube with dilation $O(1)$?

\$\$ 16. Can any N-node bounded degree planar graph be embedded one-to-one in an N-node hypercube with constant dilation? (3.77)