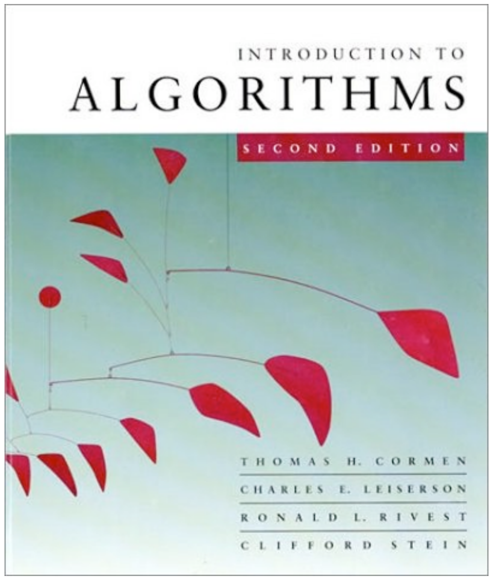
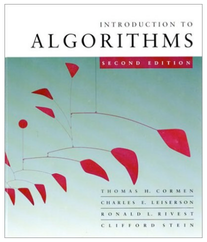
# *Introduction to Algorithms*

**6.046J/18.401J**



## *Lecture 6*

**Prof. Piotr Indyk**



## Order statistics

Select the $i$th smallest of $n$ elements (the element with ***rank*** $i$).

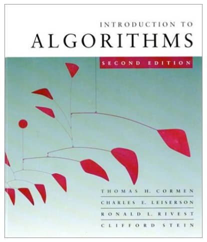
- $i = 1$: ***minimum***;
- $i = n$: ***maximum***;
- $i = \lfloor(n+1)/2\rfloor$ or $\lceil(n+1)/2\rceil$: ***median***.

***Naive algorithm***: Sort and index $i$th element.

Worst-case running time $= \Theta(n \lg n) + \Theta(1)$

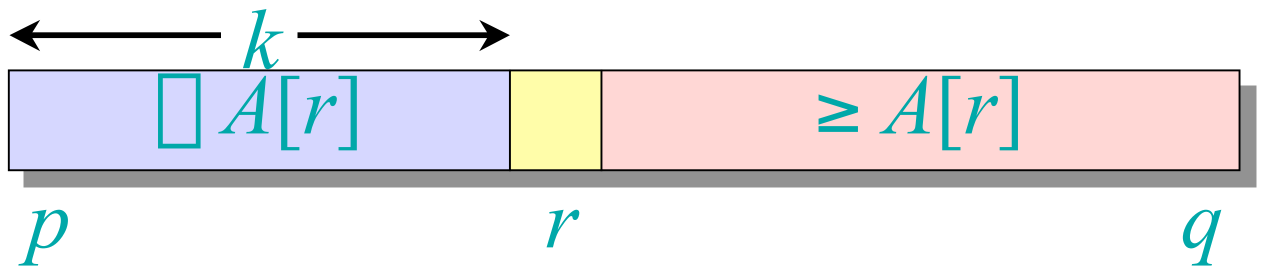
$= \Theta(n \lg n)$,

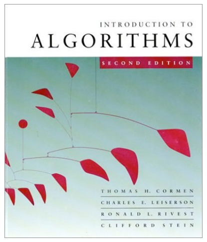
using merge sort.



# Randomized divide-and-conquer

```
RAND-SELECT($A, p, q, i$)          ⊳ $i$th smallest of $A[p \ldots q]$
    if $p = q$ then return $A[p]$
    $r \leftarrow$ RAND-PARTITION($A, p, q$)
    $k \leftarrow r - p + 1$                    ⊳ $k = \text{rank}(A[r])$
    if $i = k$ then return $A[r]$
    if $i < k$
        then return RAND-SELECT($A, p, r - 1, i$)
        else return RAND-SELECT($A, r + 1, q, i - k$)
```





## Example

Select the $i = 7$th smallest:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 6 | 10 | 13 | 5 | 8 | 3 | 2 | 11 |

$i = 7$

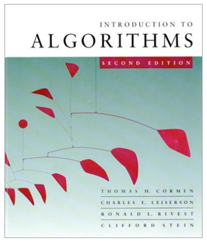
***pivot***

Partition:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 2 | 5 | 3 | 6 | 8 | 13 | 10 | 11 |

$k = 4$

Select the $7 - 4 = 3$rd smallest recursively.



## Intuition for analysis

(All our analyses today assume that all elements are distinct.)

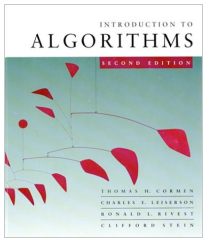
**Lucky:**

$$T(n) = T(9n/10) + \square(n) \qquad n^{\log_{10/9} 1} = n^0 = 1$$
$$= \square(n) \qquad \text{Case 3}$$

**Unlucky:**

$$T(n) = T(n-1) + \square(n) \qquad \text{arithmetic series}$$
$$= \square(n^2)$$

***Worse than sorting!***



## Analysis

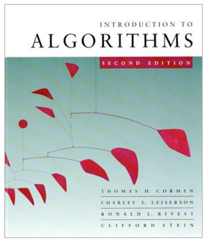
- The probability that a random pivot induces lucky partition is at least $8/10$ (Lecture 4)
- Let $t_i$ be the number of partitions performed between the $(i\text{-}1)$-th and the $i$-th lucky partition
- The total time is at most:

$$t_1\, n + t_2\,(9/10)\, n + t_3\,(9/10)^2\, n + \ldots$$

- The total ***expected*** time is at most:

$$10/8\; n + 10/8\;(9/10)\; n + 10/8\;(9/10)^2\; n + \ldots$$

$$= O(n)$$



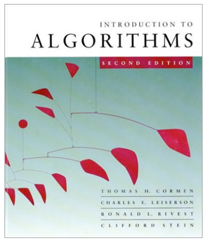
## Alternative analysis of expected time

The analysis follows that of randomized quicksort, but it's a little different.

Let $T(n)$ = the random variable for the running time of RAND-SELECT on an input of size $n$, assuming random numbers are independent.

For $k = 0, 1, \ldots, n-1$, define the ***indicator random variable***

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

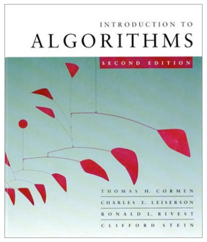


## Analysis (continued)

To obtain an upper bound, assume that the $i$th element always falls in the larger side of the partition:

$$T(n) = \begin{cases} T(\max\{0, n-1\}) + \Theta(n) & \text{if } 0 : n-1 \text{ split,} \\ T(\max\{1, n-2\}) + \Theta(n) & \text{if } 1 : n-2 \text{ split,} \\ \quad\vdots & \\ T(\max\{n-1, 0\}) + \Theta(n) & \text{if } n-1 : 0 \text{ split,} \end{cases}$$

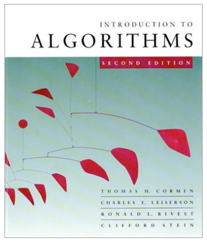
$$= \sum_{k=0}^{n-1} X_k \big(T(\max\{k, n-k-1\}) + \Theta(n)\big).$$



## Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

Take expectations of both sides.

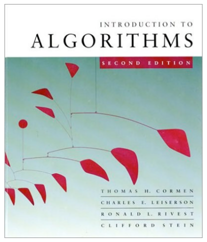


# Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \bigl(T(\max\{k, n-k-1\}) + \Theta(n)\bigr)\right]$$

$$= \sum_{k=0}^{n-1} E\bigl[X_k \bigl(T(\max\{k, n-k-1\}) + \Theta(n)\bigr)\bigr]$$

Linearity of expectation.



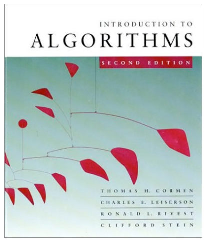
## Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \bigl(T(\max\{k, n-k-1\}) + \Theta(n)\bigr)\right]$$

$$= \sum_{k=0}^{n-1} E\bigl[X_k \bigl(T(\max\{k, n-k-1\}) + \Theta(n)\bigr)\bigr]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E[T(\max\{k, n-k-1\}) + \Theta(n)]$$

Independence of $X_k$ from other random choices.



## Calculating expectation

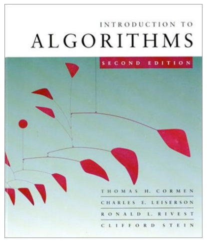
$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(\max\{k, n-k-1\}) + \Theta(n)\right]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E\left[T(\max\{k, n-k-1\})\right] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

Linearity of expectation; $E[X_k] = 1/n$ .



## Calculating expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

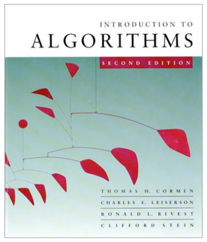
$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k\right] \cdot E\left[T(\max\{k, n-k-1\}) + \Theta(n)\right]$$

$$= \frac{1}{n}\sum_{k=0}^{n-1} E\left[T(\max\{k, n-k-1\})\right] + \frac{1}{n}\sum_{k=0}^{n-1} \Theta(n)$$

$$\le \frac{2}{n}\sum_{k=\lfloor n/2 \rfloor}^{n-1} E\left[T(k)\right] + \Theta(n)$$

Upper terms appear twice.



# Hairy recurrence

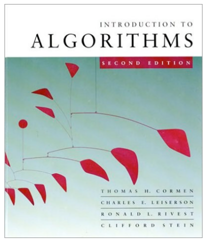
(But not quite as hairy as the quicksort one.)

$$E[T(n)] = \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)$$

**Prove:** $E[T(n)] \le cn$ for constant $c > 0$.

- The constant $c$ can be chosen large enough so that $E[T(n)] \le cn$ for the base cases.

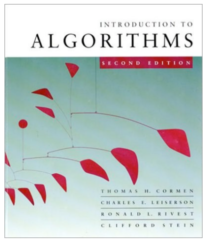
**Use fact:** $$\sum_{k=\lfloor n/2 \rfloor}^{n-1} k \le \tfrac{3}{8} n^2$$ (exercise).



## Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

Substitute inductive hypothesis.

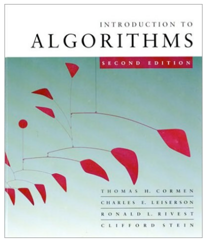


# Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

$$\le \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n)$$

Use fact.



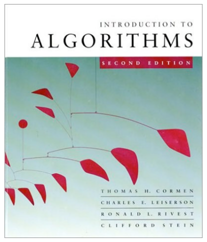
## Substitution method

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

$$\le \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n)$$

$$= cn - \left(\frac{cn}{4} - \Theta(n)\right)$$

Express as ***desired* – *residual***.



## Substitution method

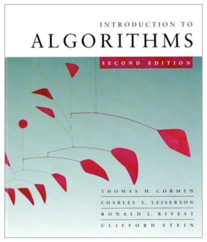
$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

$$\le \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n)$$

$$= cn - \left(\frac{cn}{4} - \Theta(n)\right)$$

$$\le cn\,,$$

if $c$ is chosen large enough so that $cn/4$ dominates the $\Theta(n)$.



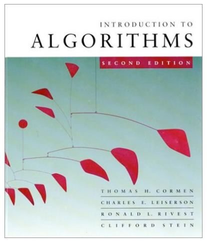
# Summary of randomized order-statistic selection

- Works fast: linear expected time.
- Excellent algorithm in practice.
- But, the worst case is ***very*** bad: □$(n^2)$.

***Q.*** Is there an algorithm that runs in linear time in the worst case?

***A.*** Yes, due to Blum, Floyd, Pratt, Rivest, and Tarjan [1973].

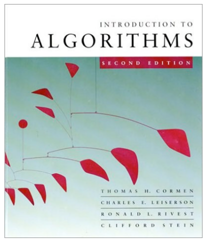
**IDEA:** Generate a good pivot recursively.



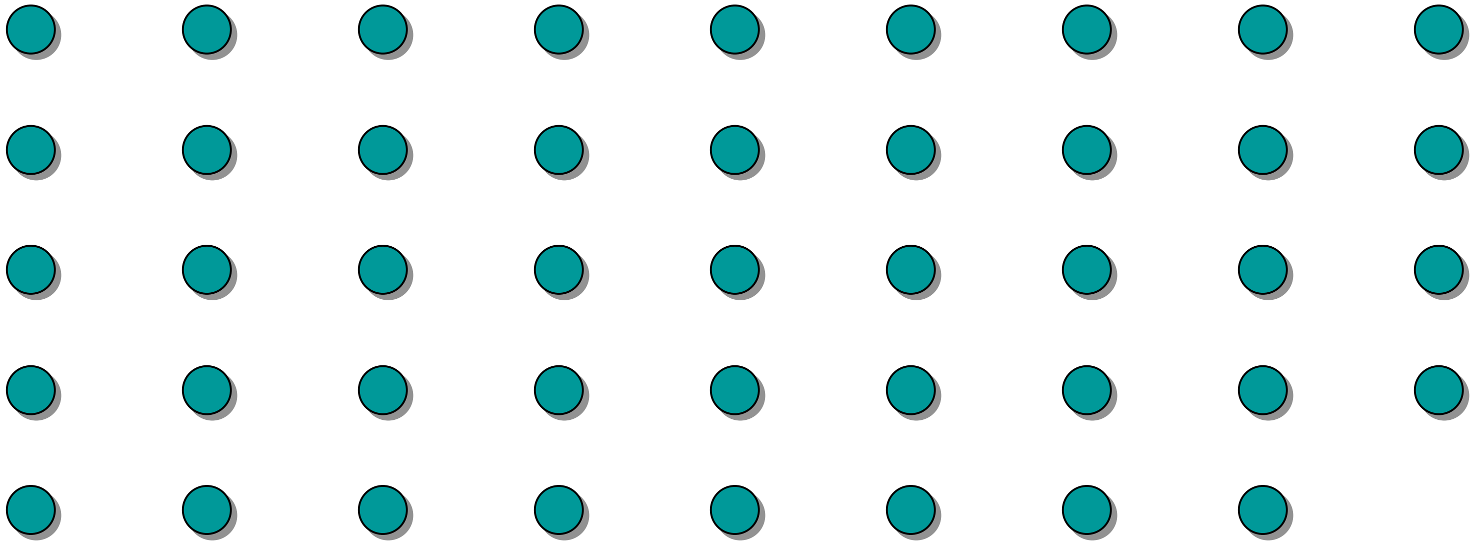
## Worst-case linear-time order statistics

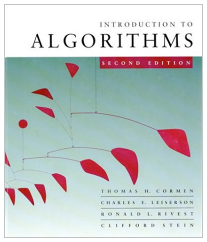
```
SELECT($i$, $n$)
1. Divide the $n$ elements into groups of 5. Find
   the median of each 5-element group by rote.
2. Recursively SELECT the median $x$ of the $\lfloor n/5 \rfloor$
   group medians to be the pivot.
3. Partition around the pivot $x$. Let $k$ = rank($x$).
4. if $i = k$ then return $x$
   elseif $i < k$
       then recursively SELECT the $i$th
           smallest element in the lower part
       else recursively SELECT the $(i–k)$th
           smallest element in the upper part
```

Same as RAND-SELECT (steps 3–4)



# Choosing the pivot





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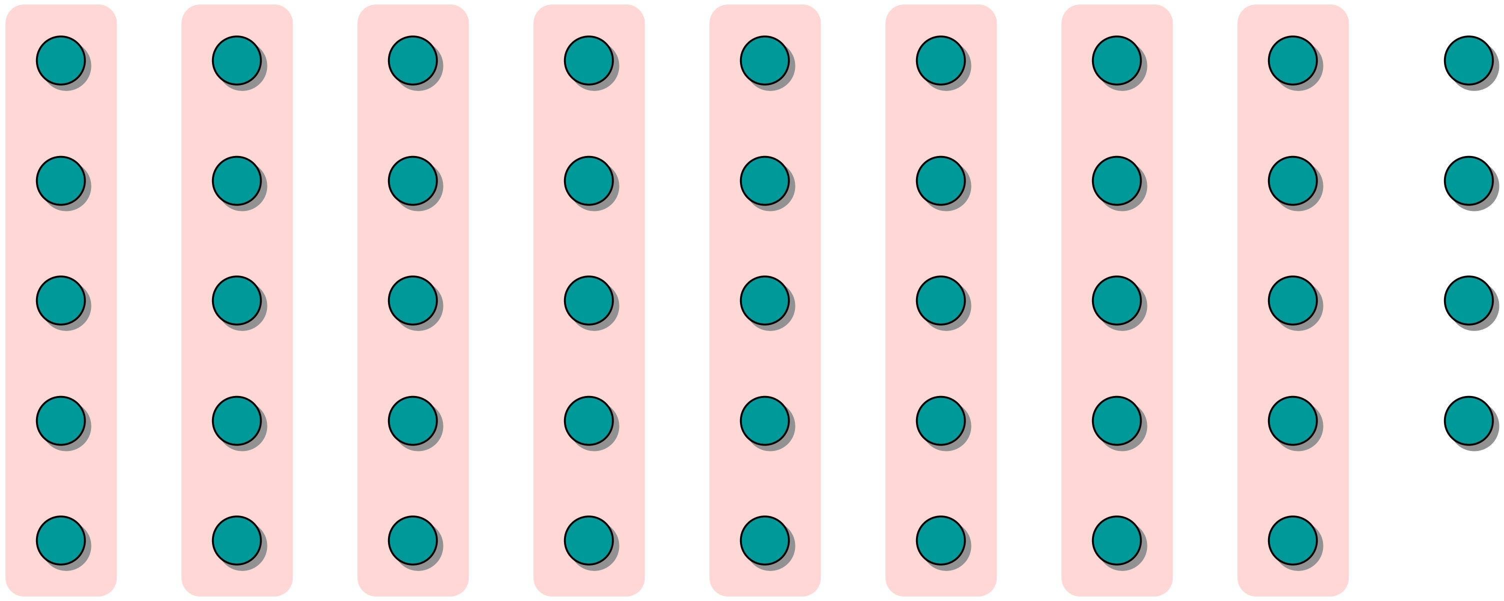
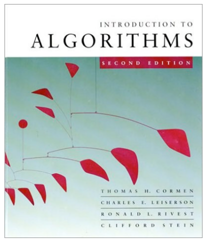
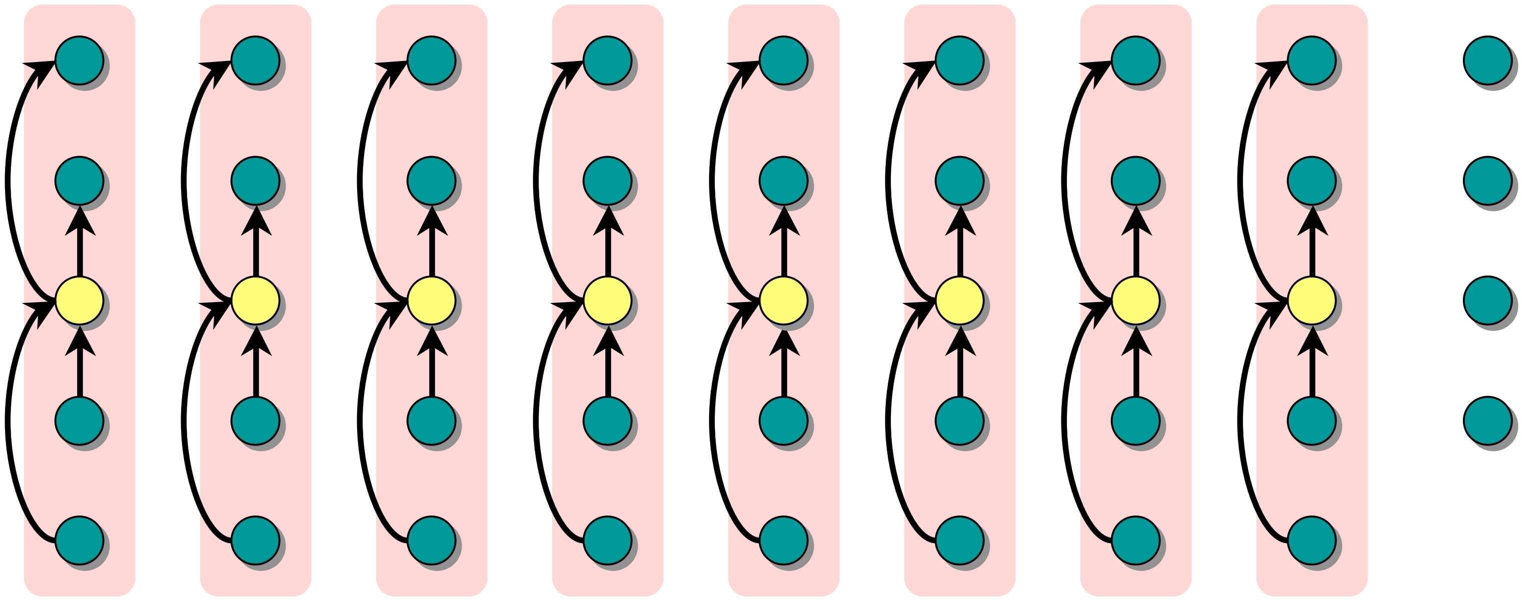


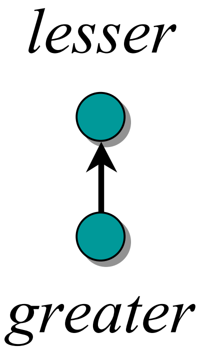
1. Divide the $n$ elements into groups of 5.

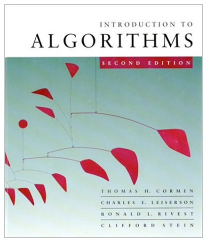


## Choosing the pivot



1. Divide the $n$ elements into groups of 5. Find the median of each 5-element group by rote.





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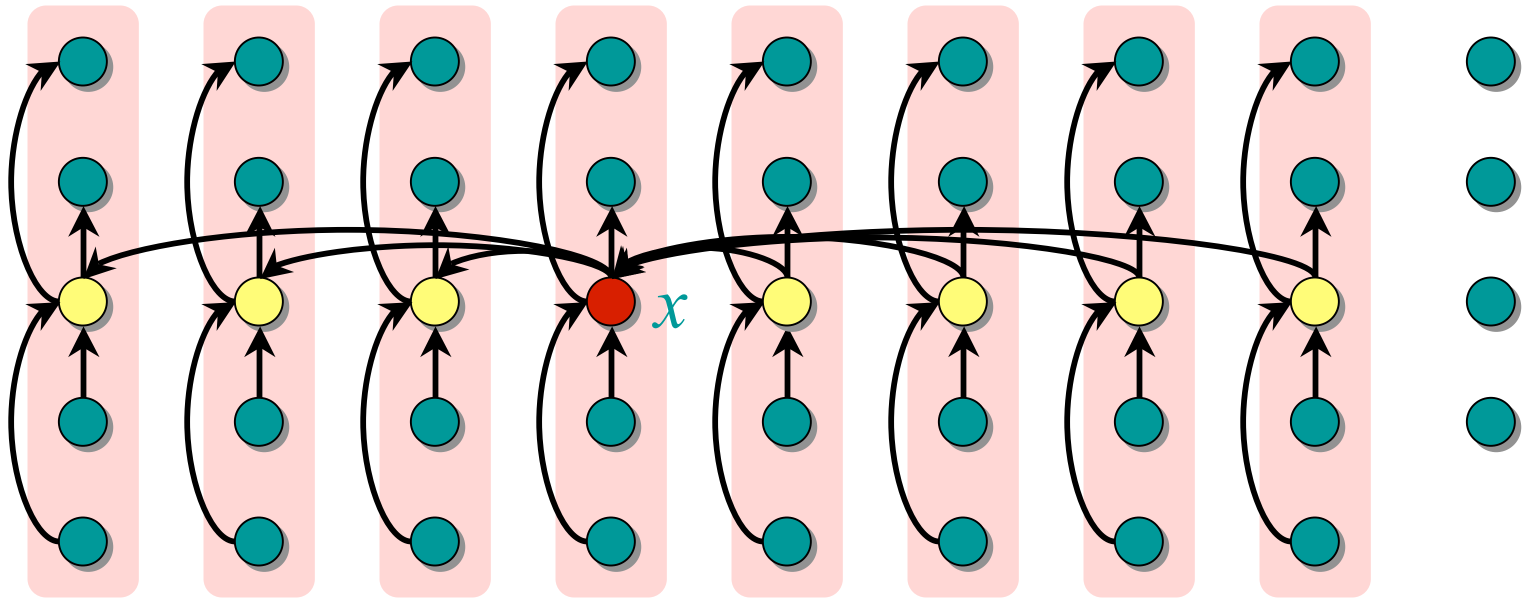
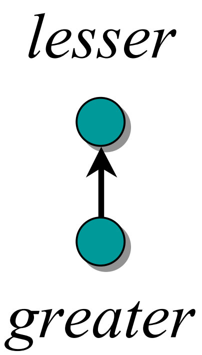
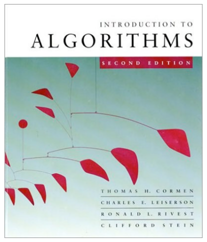
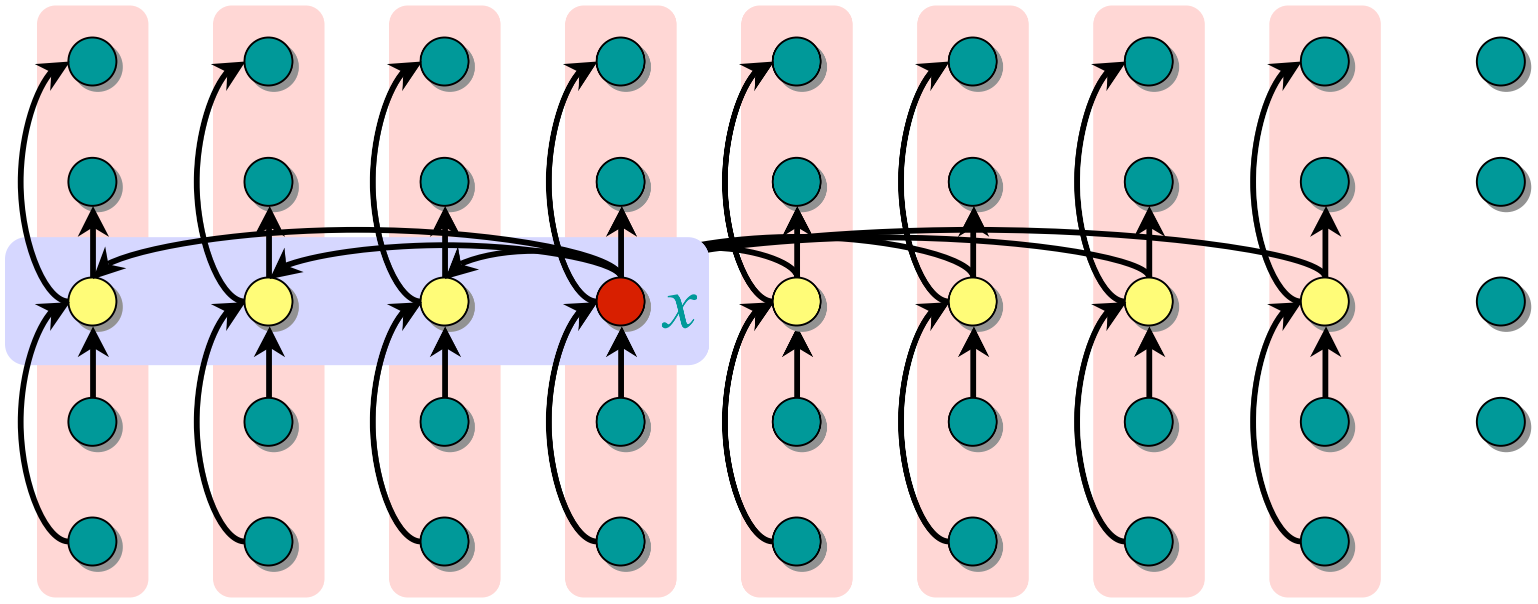


1. Divide the $n$ elements into groups of 5. Find the median of each 5-element group by rote.
2. Recursively SELECT the median $x$ of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

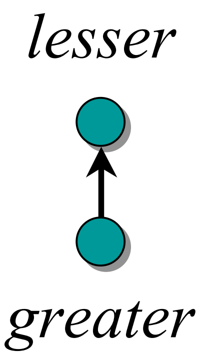


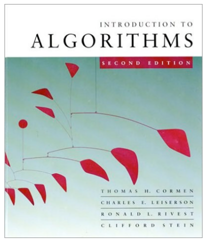


## Analysis

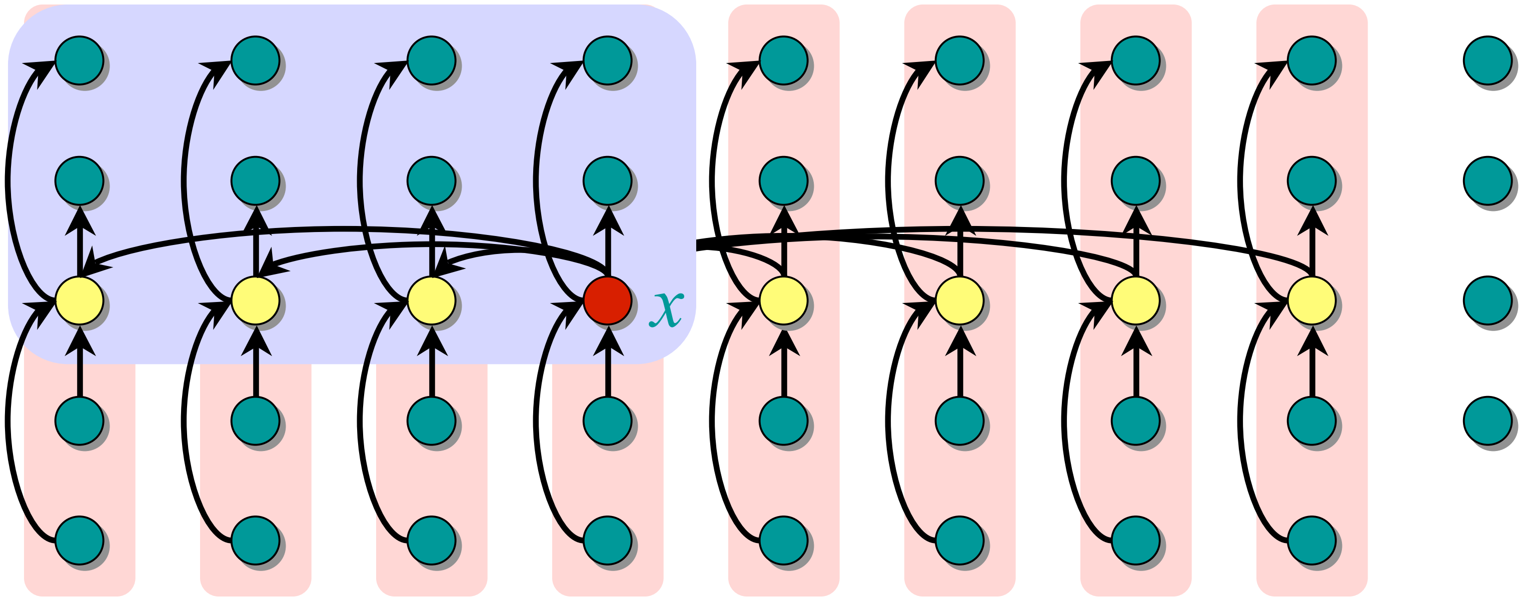


At least half the group medians are $\le x$, which is at least $\lfloor\lfloor n/5\rfloor/2\rfloor = \lfloor n/10\rfloor$ group medians.



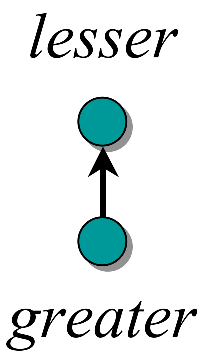


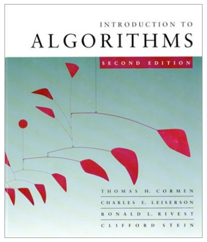
## Analysis (Assume all elements are distinct.)



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

- Therefore, at least $3 \lfloor n/10 \rfloor$ elements are $\leq x$.





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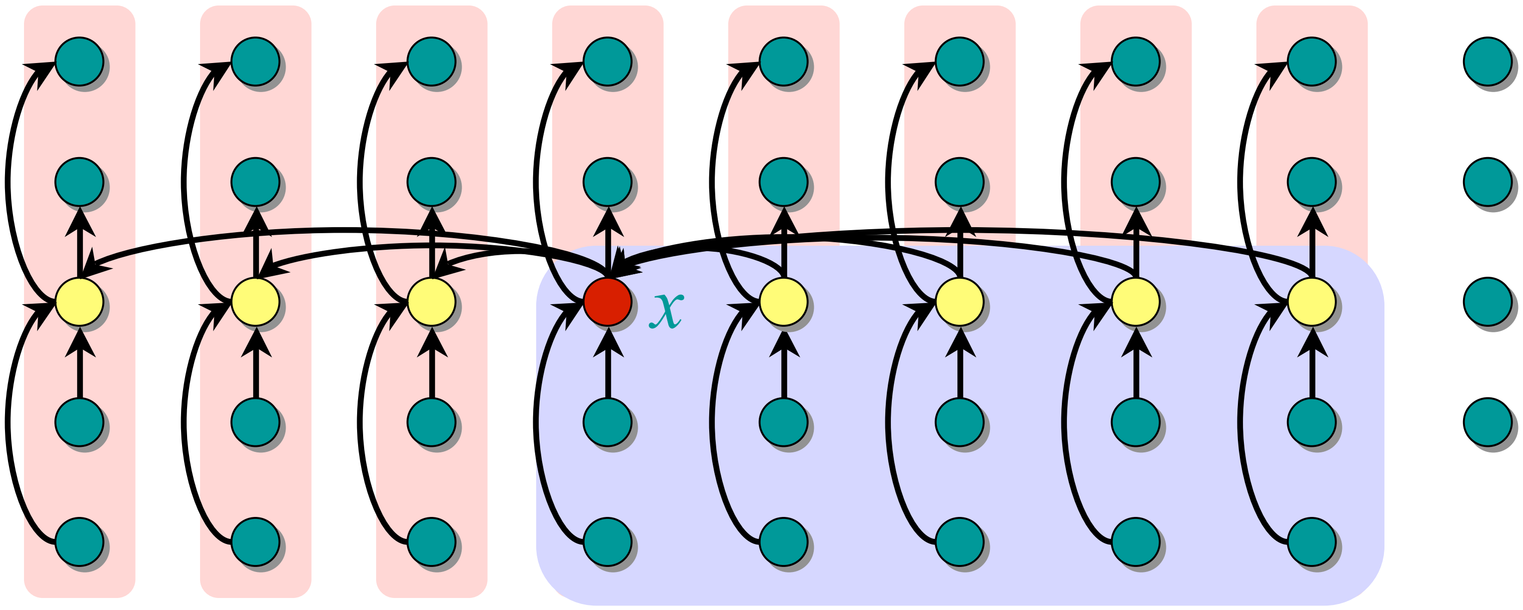
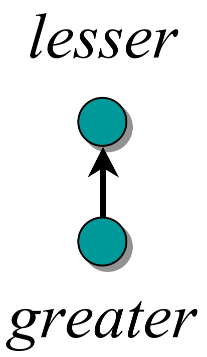
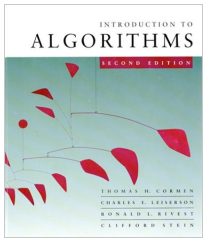


At least half the group medians are $\le x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

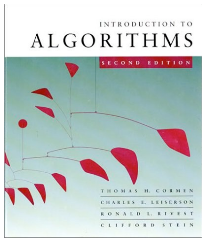
- Therefore, at least $3 \lfloor n/10 \rfloor$ elements are $\le x$.
- Similarly, at least $3 \lfloor n/10 \rfloor$ elements are $\ge x$.





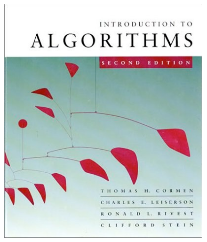
# Minor simplification

- For $n \geq 50$, we have $3\lfloor n/10 \rfloor \geq n/4$.
- Therefore, for $n \geq 50$ the recursive call to SELECT in Step 4 is executed recursively on $\leq 3n/4$ elements.
- Thus, the recurrence for running time can assume that Step 4 takes time $T(3n/4)$ in the worst case.
- For $n < 50$, we know that the worst-case time is $T(n) = \Theta(1)$.



# Developing the recurrence

```
$T(n)$         SELECT$(i, n)$
$\Theta(n)$    1. Divide the $n$ elements into groups of $5$. Find
                  the median of each $5$-element group by rote.
$T(n/5)$       2. Recursively SELECT the median $x$ of the $\lfloor n/5 \rfloor$
                  group medians to be the pivot.
$\Theta(n)$    3. Partition around the pivot $x$. Let $k = \text{rank}(x)$.
$T(3n/4)$      4. if $i = k$ then return $x$
                  elseif $i < k$
                      then recursively SELECT the $i$th
                          smallest element in the lower part
                      else recursively SELECT the $(i–k)$th
                          smallest element in the upper part
```



# Solving the recurrence

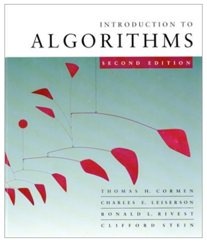
$$T(n) = T\left(\frac{1}{5}n\right) + T\left(\frac{3}{4}n\right) + \Theta(n)$$

**Substitution:**

$T(n) \le cn$

$$T(n) \le \frac{1}{5}cn + \frac{3}{4}cn + \Theta(n)$$
$$= \frac{19}{20}cn + \Theta(n)$$
$$= cn - \left(\frac{1}{20}cn - \Theta(n)\right)$$
$$\le cn \quad ,$$

if $c$ is chosen large enough to handle both the $\Theta(n)$ and the initial conditions.



## Conclusions

- Since the work at each level of recursion is a constant fraction (19/20) smaller, the work per level is a geometric series dominated by the linear work at the root.
- In practice, this algorithm runs slowly, because the constant in front of $n$ is large.
- The randomized algorithm is far more practical.

**Exercise:** *Why not divide into groups of 3?*