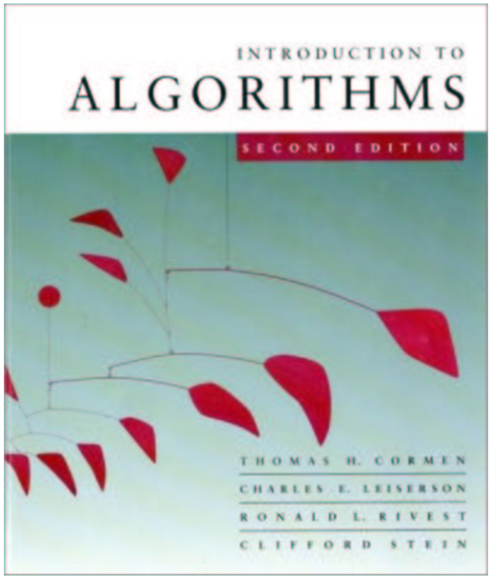
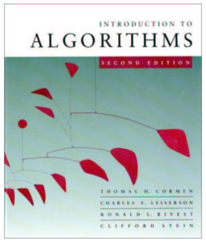
Introduction to Algorithms

6.046J/18.401



Lecture 22

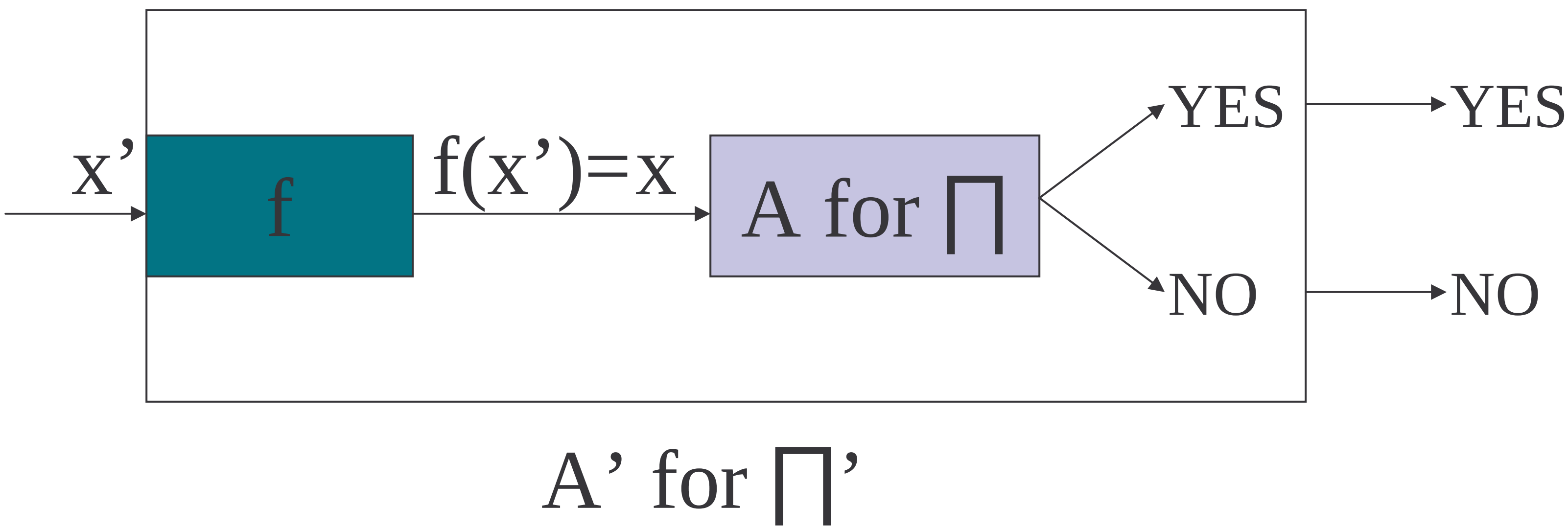
Prof. Piotr Indyk



P vs NP (Episode II)

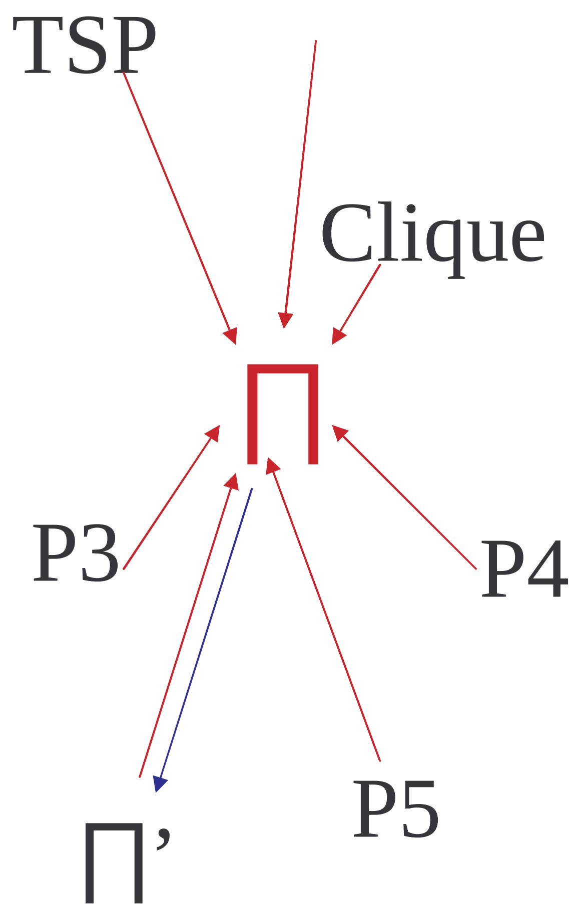
- We defined a large class of interesting problems, namely NP
  - Decision problems (YES or NO)
  - Solvable in non-deterministic polynomial time. I.e., a solution can be verified in polynomial time
- We have a way of saying that one problem is not harder than another $(\Pi' \leq \Pi)$
- Our goal: show equivalence between hard problems

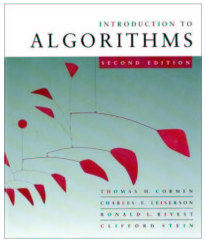
Reductions: ∏' to ∏



Showing equivalence between difficult problems

- Options:
  - Show reductions between all pairs of problems
  - Reduce the number of reductions (!) using transitivity of "≤"
  - Show that *all* problems in NP a reducible to a *fixed* ∏. To show that some problem ∏'∈NP is equivalent to all difficult problems, we only show ∏ ≤ ∏'.





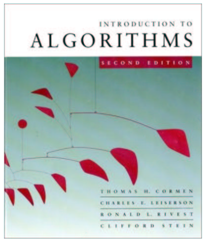
The first problem ∏

- Satisfiability problem (SAT):
  - Given: a formula φ with m clauses $C_1,\ldots,C_m$ over n variables.

    Example: $x_1 \vee x_2 \vee x_5\ ,\ x_3 \vee \neg x_5$
  - Check if there exists TRUE/FALSE assignments to the variables that makes the formula satisfiable

SAT is NP-complete

- Fact: $SAT \in NP$
- Theorem [Cook'71]: For any $\Pi' \in NP$, we have $\Pi' \le SAT$.
- Definition: A problem Π such that for any $\Pi' \in NP$ we have $\Pi' \le \Pi$, is called *NP-hard*
- Definition: An NP-hard problem that belongs to NP is called *NP-complete*
- Corollary: SAT is NP-complete.



Menu for today

SAT
↓
Clique
↓
Independent set
↓
Vertex cover

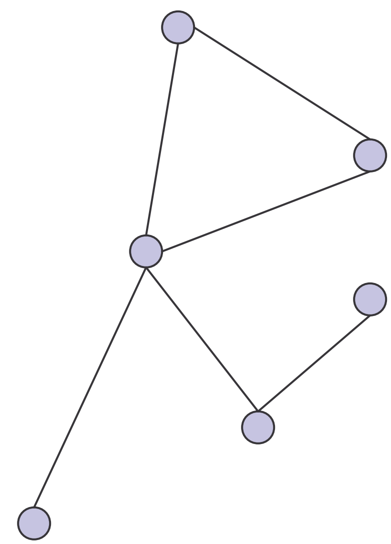
(thanks, Steve J)

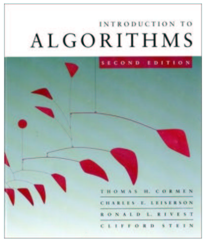
Follow from Cook's Theorem

Conclusion: all of the above problems are NP-complete

Clique again

- Clique:
  - Input: undirected graph $G=(V,E)$, K
  - Output: is there a subset C of V, $|C|\geq K$, such that every pair of vertices in C has an edge between them



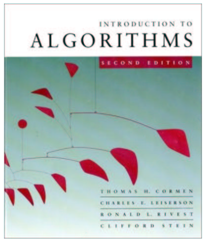


SAT ≤ Clique

- Given a SAT formula $\varphi = C_1, \ldots, C_m$ over $x_1, \ldots, x_n$, we need to produce $G=(V,E)$ and K, such that φ satisfiable iff G has a clique of size $\geq K$.
- Notation: a literal is either x_i or $\neg x_i$

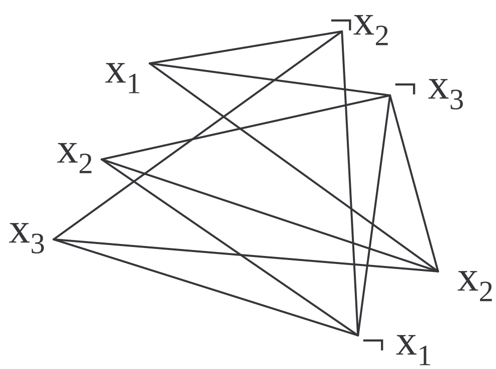
SAT ≤ Clique reduction

- For each literal t occurring in φ, create a vertex v_t
- Create an edge $v_t - v_{t'}$ iff:
  - t and t' are not in the same clause, and
  - t is not the negation of t'



SAT ≤ Clique example

- Formula: $x_1 \vee x_2 \vee x_3$, $\neg x_2 \vee \neg x_3$, $\neg x_1 \vee x_2$
- Graph:

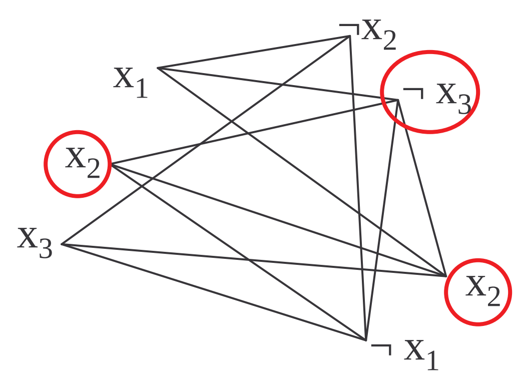


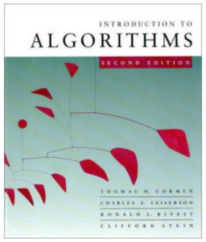
- Claim: φ satisfiable iff G has a clique of size $\geq m$

Proof

- "→" part:
  - Take any assignment that satisfies φ.

    E.g., x_1=F, x_2=T, x_3=F
  - Let the set C contain one satisfied literal per clause
  - C is a clique

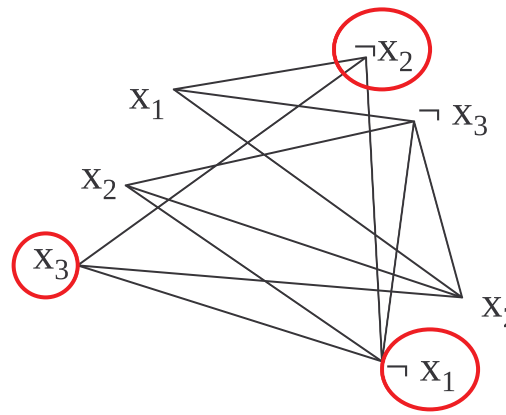


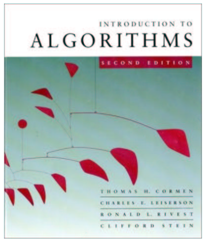


Proof

- "←" part:
  - Take any clique C of size $\geq m$ (i.e., $= m$)
  - Create a set of equations that satisfies selected literals.

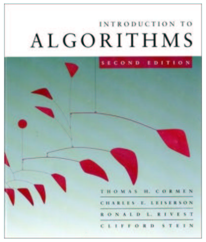
    E.g., x_3=T, x_2=F, x_1=F
  - The set of equations is consistent and the solution satisfies φ





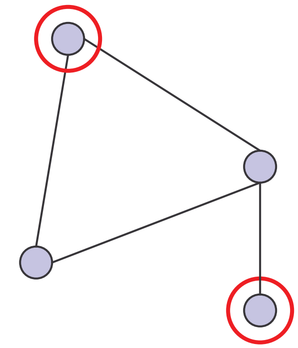
Altogether

- We constructed a reduction that maps:
  - YES inputs to SAT to YES inputs to Clique
  - NO inputs to SAT to NO inputs to Clique
- The reduction works in poly time
- Therefore, SAT $\leq$ Clique $\rightarrow$ Clique NP-hard
- Clique is in NP $\rightarrow$ Clique is NP-complete



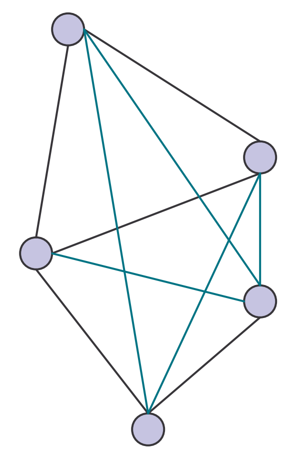
Independent set (IS)

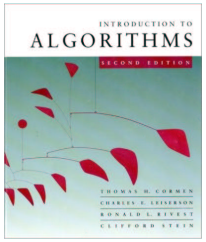
- Input: undirected graph $G=(V,E)$
- Output: is there a subset S of V, $|S| \geq K$ such that no pair of vertices in S has an edge between them



Clique ≤ IS

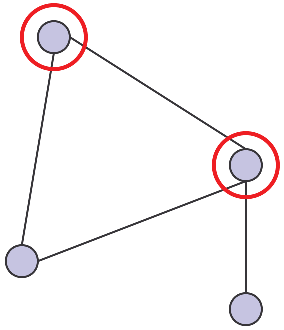
- Given an input G=(V,E), K to Clique, need to construct an input G'=(V',E'), K' to IS, such that G has clique of size ≥K iff G' has IS of size ≥K.
- Construction: K'=K, V'=V, $E'=\overline{E}$
- Reason: C is a clique in G iff it is an IS in G's complement.





Vertex cover (VC)

- Input: undirected graph G=(V,E)
- Output: is there a subset C of V, |C| ≤ K, such that each edge in E is incident to at least one vertex in C.



IS ≤ VC

- Given an input $G=(V,E)$, K to IS, need to construct an input $G'=(V',E')$, K' to VC, such that G has an IS of size $\geq K$ iff G' has VC of size $\leq K'$.
- Construction: $V'=V$, $E'=E$, $K'=|V|-K$
- Reason: S is an IS in G iff $V-S$ is a VC in G.