Massachusetts Institute of Technology
6.046J/18.410J: Introduction to Algorithms
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Handout 5
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Some commonly used series

The following series are used commonly in the design and analysis of algorithms. Each is quite useful but can seem frustratingly mysterious without knowledge of a simple proof. You are encouraged to prove them for yourself. The sums are arranged in a particular order: often the proof of an equality depends on the preceding equalities or their proofs.

Hints for each series are included on the back of this sheet. When looking for a hint, be careful not to read hints for other problems (which would spoil your fun)! You can do this by covering the hint sheet with two pieces of paper and revealing only the desired hint.

Enjoy!

1. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

2. $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

3. $\displaystyle\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$

4. $\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \qquad (|x| < 1)$

5. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$

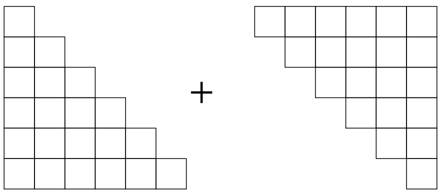
6. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \cos x$

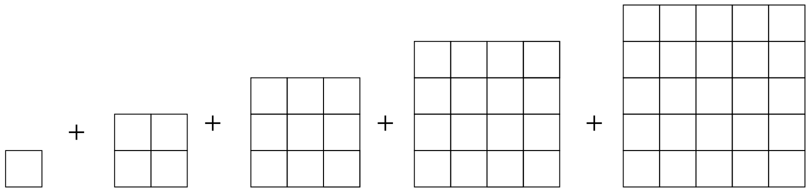
7. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sin x$

8. $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k} = \ln\left(\frac{1}{1-x}\right)$

9. $\displaystyle\sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$

Hints

1. 

2. 

   Each block is made up of i columns for $i = 1, 2, \ldots, n$. Add up the first column of block j for $j = 1, \ldots, n$. Add up the second column of block j for $j = 2, \ldots, n$, etc. You get
$$\sum_{k=1}^{n} k^2 = \sum_{i=1}^{n}\sum_{j=i}^{n} j.$$

3. Multiply both sides by $x - 1$.

4. Take the limit in the previous problem.

5. $$\begin{aligned} \sum_{k=0}^{\infty} a_k x^k &= F(x) \\ \Rightarrow \quad \sum_{k=1}^{\infty} k a_k x^{k-1} &= F'(x) \\ \Rightarrow \quad \sum_{k=1}^{\infty} k a_k x^k &= xF'(x). \end{aligned}$$

   If $F(x) = e^x$ then $F(x) = F'(x)$.

6. $e^{i\theta} = \cos\theta + i\sin\theta$, where $i = \sqrt{-1}$.

7. Same hint as 6.

8. Same hint as 5.

9. What is the integral of $1/x$ from 1 to n? Plot $1/x$ from 1 to n and on the same graph draw the sum of $1/k$ from $k = 1 \ldots n$. Upper bound this sum by an integral of $1/x$ and lower bound it by another integral of $1/x$ (different ranges).