Lecture 25: May 14, 2002

Today: Approximation algorithms

Coping with NP-hardness

- "Efficient" exponential time algorithms for small inputs
(e.g., $(100/99)^n$ time is not bad for $n \leq 1000$)
- Polynomial time algorithms for some (e.g., random) inputs
- Polynomial time algorithms for all inputs, but *incorrect* (popular approach, but not advised)
- Polynomial time algorithms for all inputs, but *approximate*

Approximation algorithms

Consider *minimization* problem Π. An algorithm A for Π is *ρ-approximate*, if for any input, the cost C_A of the solution it produces is at most ρ times the cost C^* of the optimal solution.

E.g., an algorithm which produces TSP tour at most 50% longer than the optimal one, is 1.5-approximate.

Similar definition for *maximization* problems.

We are only interested in polynomial time approximation algorithms.

Examples of approximation algorithms

Are there any approximation algorithms ?

Yes, almost every NP-hard problem has a corresponding approximation algorithm. In particular:

- TSP: has a 2-approximation algorithm (can be improved to 1.5).
- Set Cover: has $O(\log n)$-approximation algorithm.
- Vertex Cover: has 2-approximation algorithm.

TSP

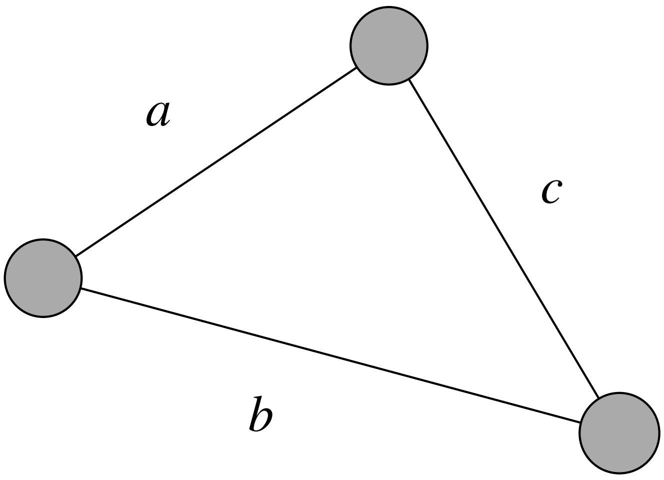
- Input: Undirected *complete* graph with lengths $c(u, v)$ on edges.
- Output: Smallest tour that visits every vertex exactly once.
- Problem is NP-hard:
  - Hamiltonian cycle (HC): given an unweighted graph, is there *any* tour which visits any vertex exactly once ?
  - HC is NP-hard (CLRS)
  - HC is poly time reducible to TSP:
    - if $\{u, v\}$ in the graph $\Rightarrow c(u, v) = 1$
    - if $\{u, v\}$ not in the graph $\Rightarrow c(u, v) = W$
  - HC exists iff there is a TSP tour with cost $= n$
  - No HC $\Rightarrow$ any TSP tour has cost $\geq W + (n - 1)$

Approximating TSP

- But this means there is no poly time W/n-approximation algorithm (otherwise we have a poly time algorithm for HC)
- W can be very large $\Rightarrow$ panic ?
- Suppose weights satisfy **triangle inequality:**

$$c(u,w) \leq c(u,v) + c(v,w)$$

for all $u, v, w \in V$



$a + b \leq c$

- TSP is still NP-complete (take $W = 2$)
- However, it *can* be solved approximately

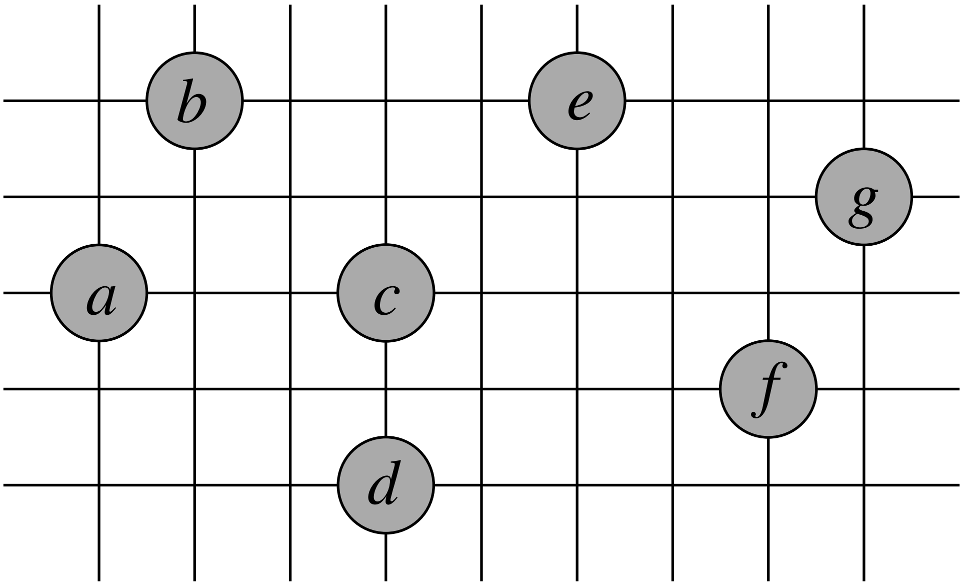
TSP Approximation Algorithm

APPROX-TSP-TOUR

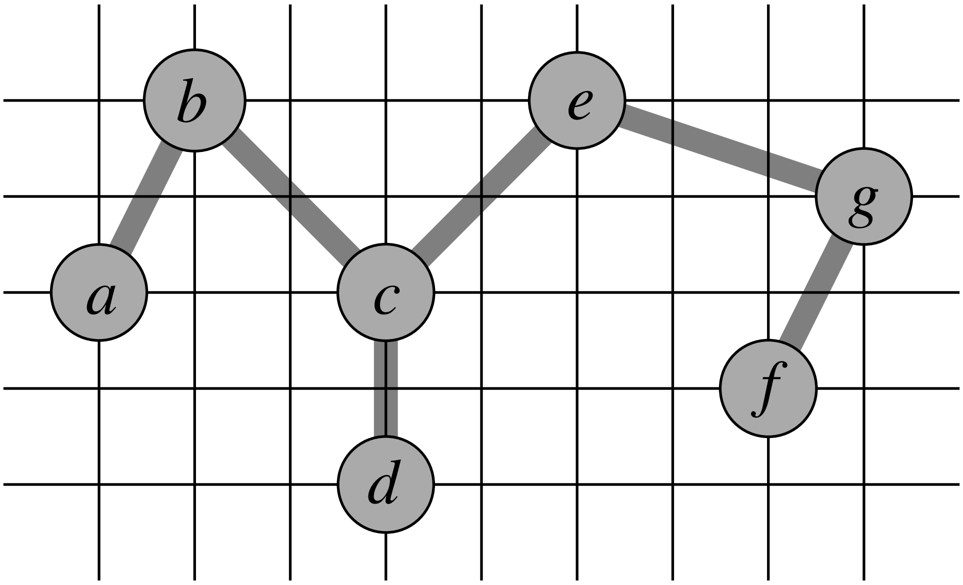
1. Build a minimum spanning tree T for G
2. Let L be the list of vertices visited in preorder tree walk of T
3. **return** a tour H that visits vertices in the order L

Example

- Find shortest tour for:

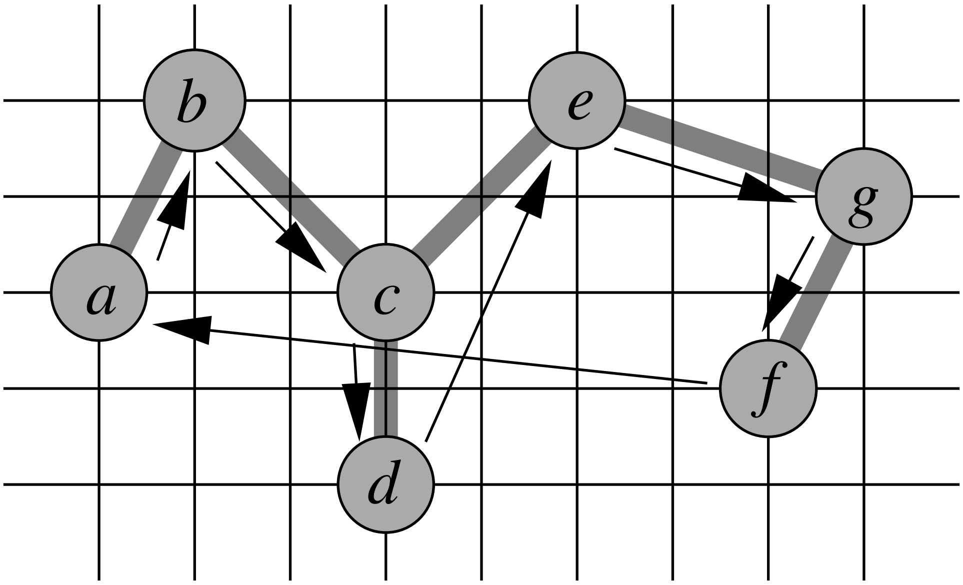


- Find MST.

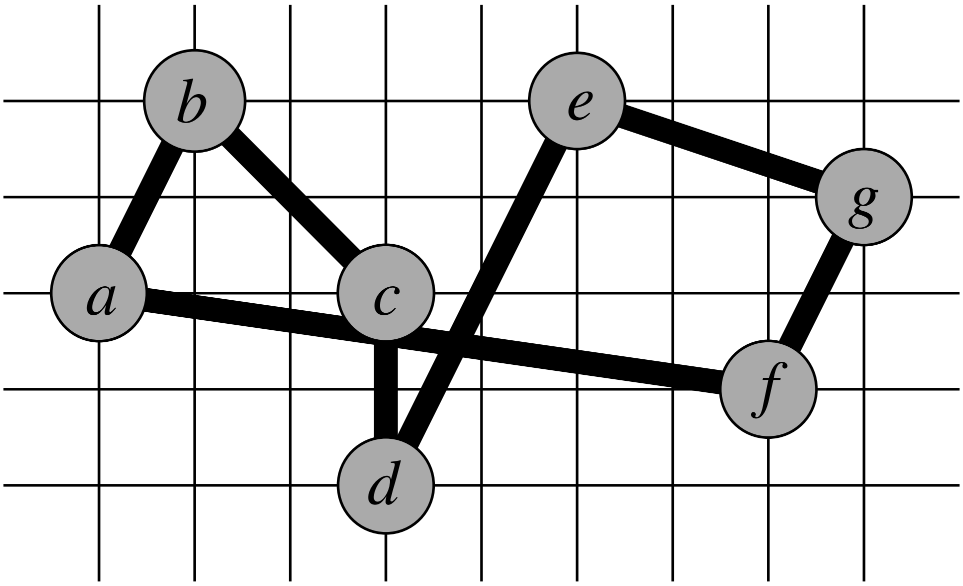


Example

- Short-cutting walk of MST

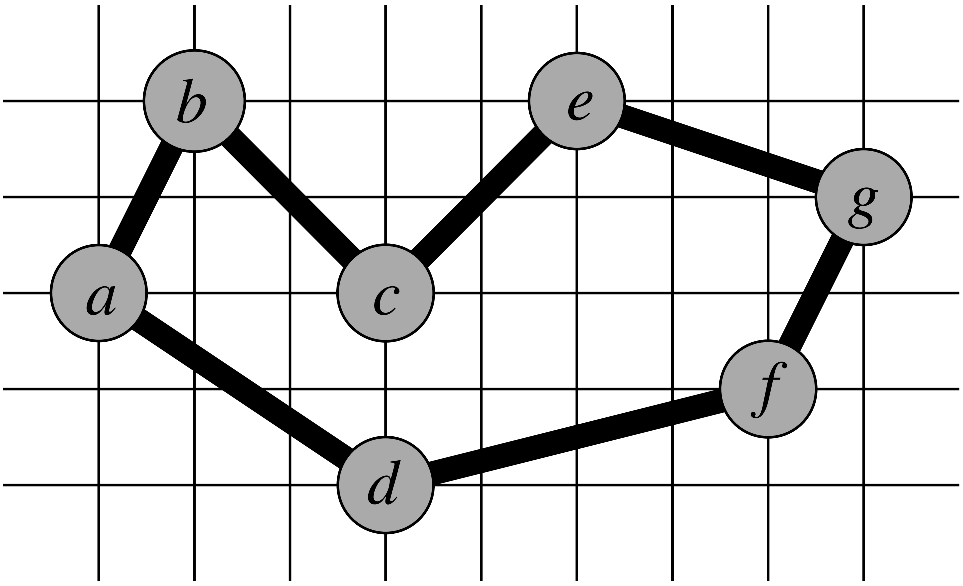


- Yielding tour:



- Total distance $\approx$ 24.00 units.

Optimal Solution



- Total distance $\approx$ 20.44 units.

Approximation factor

Theorem: APPROX-TSP-TOUR with triangle inequality has ratio bound 2.

Proof:

- $H^* =$ optimal tour for G.
- T is a MST for $G \Rightarrow c(T) \leq c(H^*)$.
- $W =$ full walk of $T \Rightarrow c(W) = 2c(T)$ (each MST edge is taken twice)
- H is the tour obtained by shortcutting T. By triangle inequality, $c(H) \leq c(W)$
- $c(H) \leq c(W) \leq 2c(T) \leq 2c(H^*)$

Set cover

Set cover (SC):

- Input: a family $\mathcal{S}$ of sets $S_1 \dots S_n \subset U$, $|U| = m$, such that $\cup_i S_i = U$.
- Output: A set $I \subset \{1 \dots n\}$ of smallest size, such that
$$\cup_{i \in I} S_i = U$$

Very natural problem, e.g.:

- U- the set of tasks
- S_i - the set of tasks which can be performed by person i
- Want to find the minimal team which can perform all tasks

Unfortunately, problem is NP-complete
(previous lecture).

Greedy algorithm

Repeat until all elements are covered:

- Choose a new set S_i which contains largest number of yet uncovered elements
- Add i to I
- Mark all elements from S_i as covered

Natural algorithm, but can be fooled
(e.g., consider sets $\{1,2\},\{3,4\},\{5,6\},\{1,3,5\}$).

Nevertheless, it is an $\ln m$-aproximation algorithm

Proof

Let k be the size of the minimal cover.

Fact: At any iteration of the algorithm, there is at least one set S_i which covers $\geq 1/k$ fraction of yet uncovered elements.

Note 1: Such S_i cannot be in the current cover (i.e., $i \notin I$), since by definition no S_i included in the current cover contains any uncovered elements (i.e., all such elements are marked as covered).

Note 2: The above note is not relevant for the correctness of the algorithm. It was included since a few students asked about it. If it confuses you, just ignore it.

Proof: If each set covered $< 1/k$ fraction of (yet uncovered) elements, then there would be no way to cover U using only k sets.

$\ln m$-approximation

Let u_i be the number of uncovered elements after the ith step. We know that

$$u_{i+1} \leq (1 - 1/k)u_i$$

Initially, $u_0 = m$, thus

$$u_i \leq (1 - 1/k)^i m$$

Therefore, for $i = k \ln m$ we have

$$u_i \leq (1 - 1/k)^{k \ln m} m \leq e^{-\ln m} m = 1$$

and thus after $k \ln m + 1$ steps the algorithm covers all elements and stops.

Thus, greedy is $(\ln m + 1/k)$-approximate (we skip the $1/k$ term for simplicity).

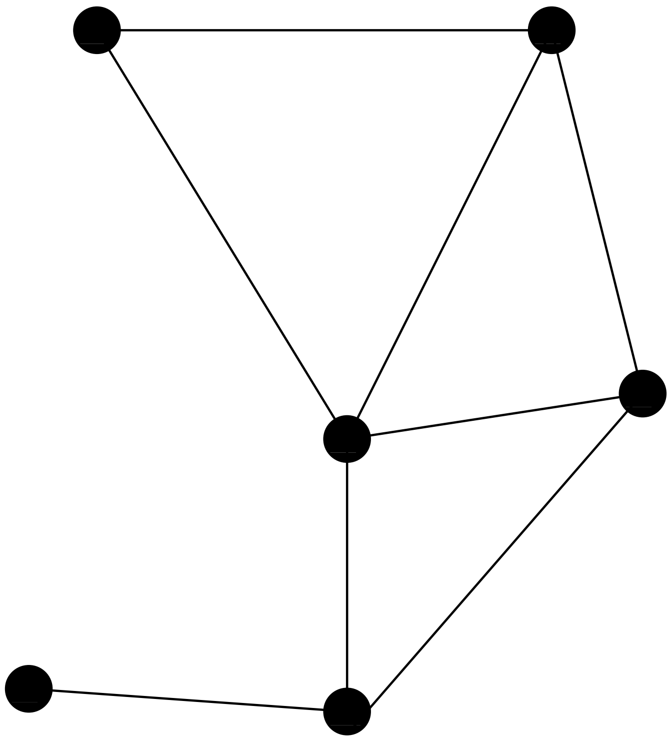
Set cover summary

- Greedy gives $\ln m$-approximation
- Cool fact: Feige'95 showed that no $(1-\epsilon)\ln m$-approximation algorithm exists, unless P=NP
- Not-so-cool (but useful) fact: greedy is in fact

$$(\ln m - \ln\ln m + 0.78) - \text{approximate}$$

Vertex cover

- Input: Undirected graph (V, E).
- Output: A set $C \subseteq V$ of minimum size such that every edge in E has at least one endpoint in C ?
- Special case of set cover, when every element is contained in only two sets

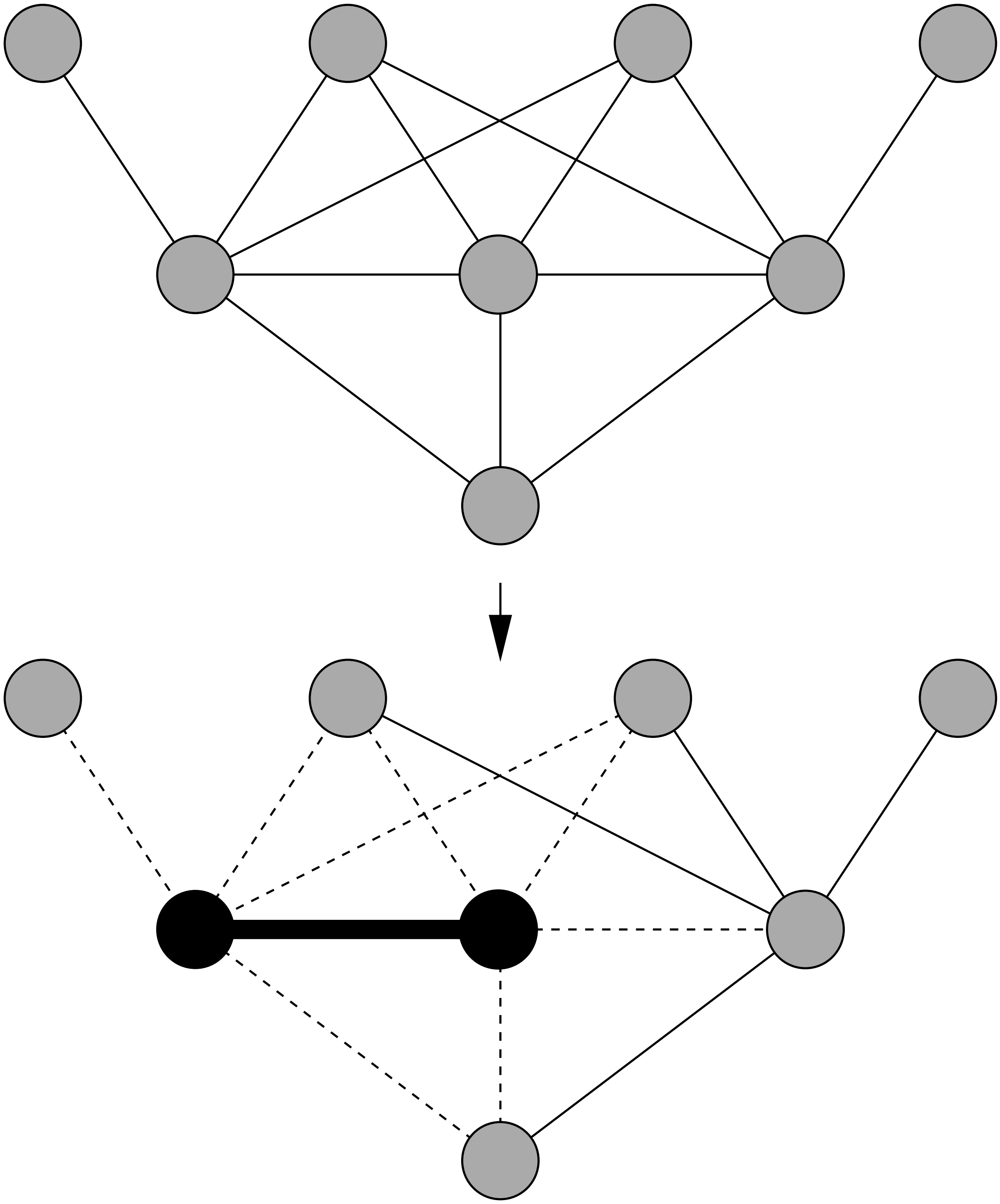


Approximation Algorithm

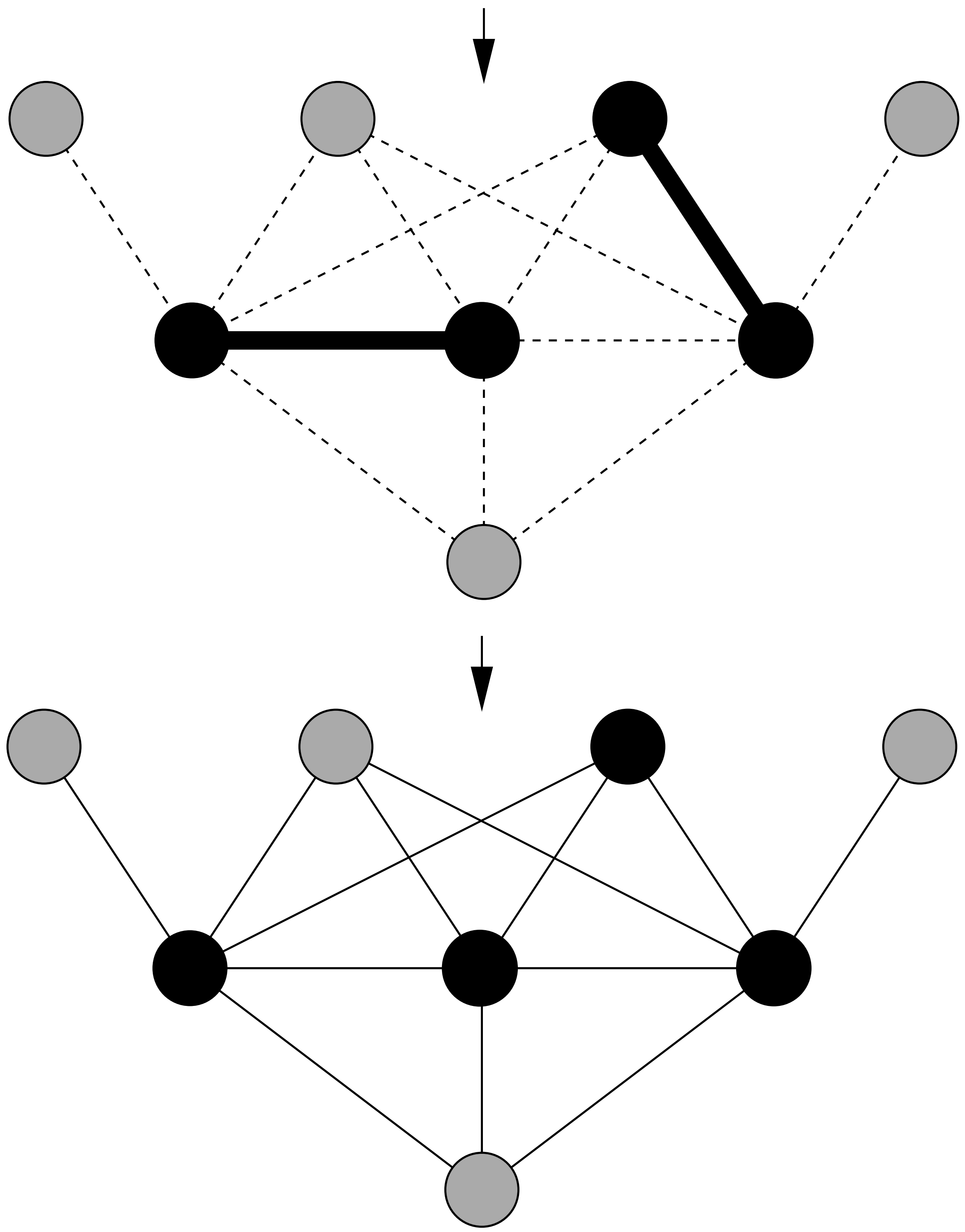
APPROX-VERTEX-COVER(G)

$C \leftarrow \emptyset$
$E' \leftarrow E[G]$
while $E' \neq \emptyset$
    do let $\{u, v\}$ be an arbitrary edge of E'
        $C \leftarrow C \cup \{u, v\}$
        remove from E' every edge incident to
            either u or v
return C

Vertex Cover Approximation Example



Vertex Cover Approximation Example Cont.



Analysis of Vertex Cover Approximation

- **Correctness**
  - Only remove "covered" edges from E'.
  - APPROX-VERTEX-COVER returns a vertex cover.
- **Running Time** is $O(V + E)$.

Further Analysis

- **Theorem** APPROX-VERTEX-COVER has ratio bound 2.
- **Proof**
  - $A = \{$edges picked in line 4$\}$ (A is a set).
  - No two edges in A share an endpoint.
  - $|C| = 2\,|A|$.
  - Optimal cover, C^* must include at least one endpoint for each edge in A.
  - $|A| \leq |C^*|$.
  - Conclude that $|C| \leq 2\,|C^*|$;
    that is, size of approximate cover is
    at worst twice size of optimal cover!