Lecture 24: May 9, 2002

Today:

- NP-completeness continued (connecting more things)

In the previous episode

- NP - problems whose solutions are verifiable in polynomial time
- Poly time reductions
- Any problem in NP can be reduced to SAT, SAT in NP $\Rightarrow$ SAT NP-complete
- SAT $\rightarrow$ 3SAT, 3SAT in NP $\Rightarrow$ 3SAT NP-complete

Today's menu

- SAT $\to$ Clique
- Clique $\to$ Independent Set
- Independent Set $\to$ Vertex Cover
- Vertex Cover $\to$ Set Cover
- Set Cover $\to$ Dominating Set

Therefore, all of them are NP-hard (and NP-complete).

SAT $\to$ Clique

Clique:

- Input: Undirected graph (V, E), $K > 0$.
- Output: Is there $C \subseteq V$ of size $\geq K$ such that every pair of vertices in C has an edge between them

SAT:

- Input: n Boolean variables $x_1, \ldots, x_n$ and a formula ϕ with m clauses, e.g.,

$$x_1 \vee x_2 \vee x_5 \ , \ \ x_1 \vee \neg x_2, ...$$

- Goal: Check if there exists a TRUE/FALSE assignments of variables such that every clause is satisfied, i.e., every clause above has one term set to TRUE.

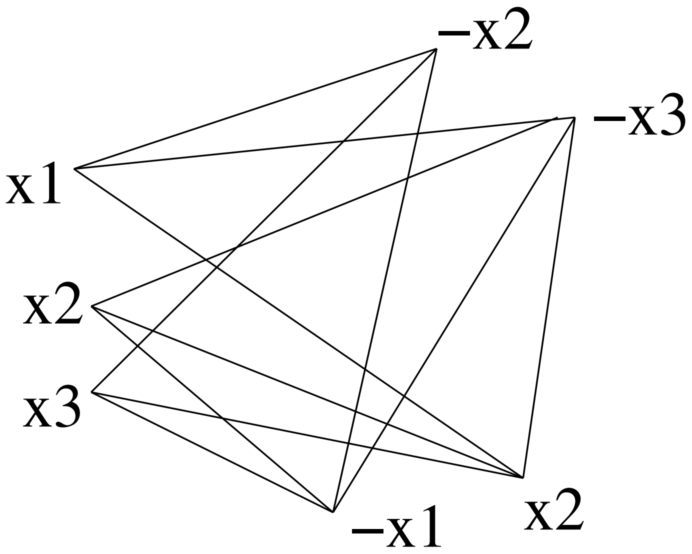
Our goal: given a SAT formula $\phi = C_1 \ldots C_m$, produce $G = (V, E)$ and K, such that ϕ satisfiable if and only if G has clique of size $\geq K$.

Reduction

Glossary: a *literal* is either x_i or $\neg x_i$.
Reduction:

- For each literal l_j in ϕ, create a vertex v_j
- Create an edge $v_j - v_{j'}$ if
  - $v_j, v_{j'}$ do not correspond to literals from the same clause
  - l_j is *not* a negation of $l_{j'}$

E.g., the formula $x_1 \vee x_2 \vee x_3$, $\neg x_2 \vee \neg x_3, \neg x_1 \vee x_2$ is transformed into graph



Claim: ϕ satisfiable if and only if clique $\geq m$.

Independent set

Independent set (IS):

- Input: Undirected graph (V, E), $K > 0$.
- Output: Is there $S \subseteq V$ of size $\geq K$ such that every pair of vertices in S has *no* edge between them

Clique for $(V, E) \to$ IS for (V', E'):

- Keep the same vertex set, i.e., $V' = V$
- Take E' to be the *complement* of E

Claim: any clique C in (V, E) is an independent set in (V', E').

Corollary: (V, E) has clique $\geq K$ if and only if (V', E') has independent set $\geq K$.

Vertex cover

Vertex cover (VC):

- Input: Undirected graph (V, E), $L > 0$.
- Output: Is there $C \subseteq V$ of size $\leq L$ such that every edge in E has at least one endpoint in C ?

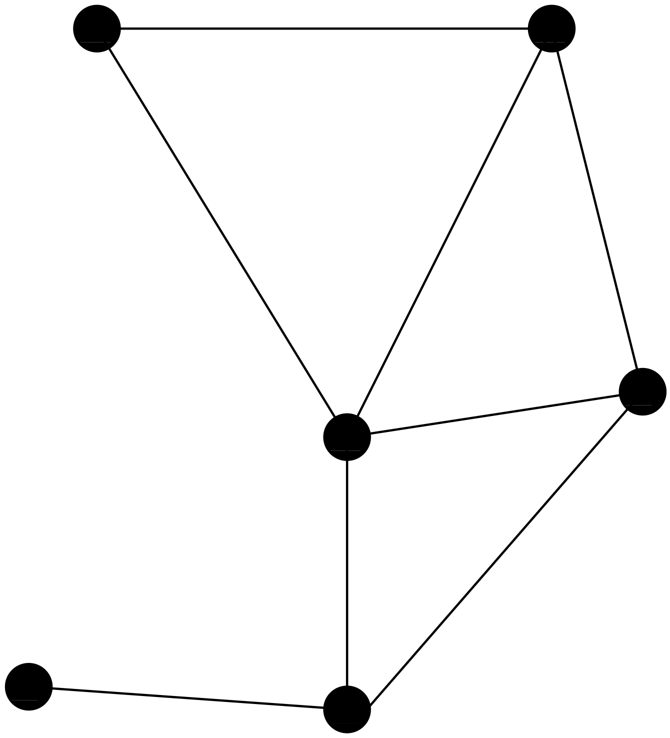
IS for $(V, E) \rightarrow$ VC for (V', E'):

- Keep the same vertex set $(V' = V)$
- Keep the same edge set $(E' = E)$
- Set $L = n - K$

Claim: S is an independent set in (V, E) if and only if the *complement* of S is a vertex cover for (V', E').

Corollary: (V, E) has independent set $\geq K$ if and only if (V', E') has vertex cover $\leq n - K$.

Picture



Set cover

Set cover (SC):

- Input: a family $\mathcal{S}$ of sets $S_1 \dots S_n \subset U$, $L > 0$.
- Output: Is there $I \subset \{1 \dots n\}$, of size $\leq L$ such that

$$\cup_{i \in I} S_i = U \qquad ?$$

VC for $(V, E) \rightarrow$ SC for $\mathcal{S}, U$:

- Set $U = E$
- For each vertex $v \in V$, set S_v to be the set of edges incident to v

Claim: C is a vertex cover for (V, E) if and only if $I = C$ covers E.

Dominating set

Dominating set (DS):

- Input: a graph $G = (V, E)$, $L > 0$.
- Output: Is there $S \subset V$, of size $\leq L$, which *dominates* V, i.e., such that any vertex is either in S is adjacent to a vertex in S ?

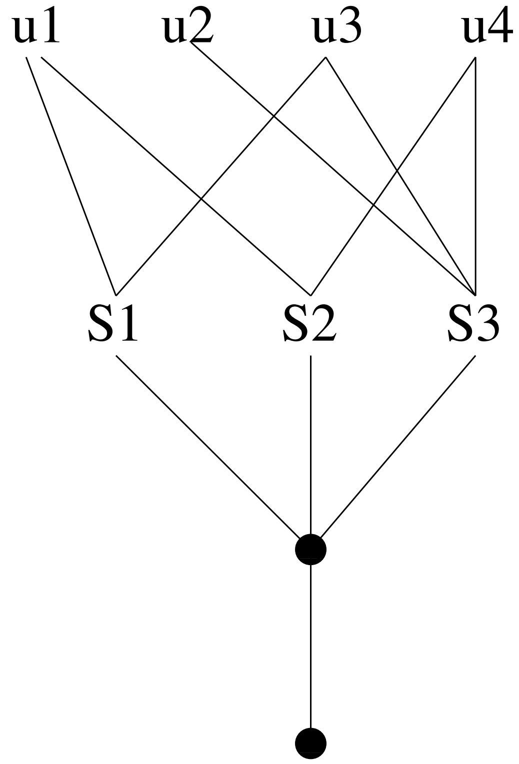
SC for $\mathcal{S}, U \rightarrow$ DS for (V, E).

SC to DS

Proof by example:

- Universe $U = \{u_1, u_2, u_3, u_4\}$
- Sets $\{u_1, u_3\}, \{u_1, u_4\}, \{u_2, u_3, u_4\}$

We create a graph (V, E) as follows:

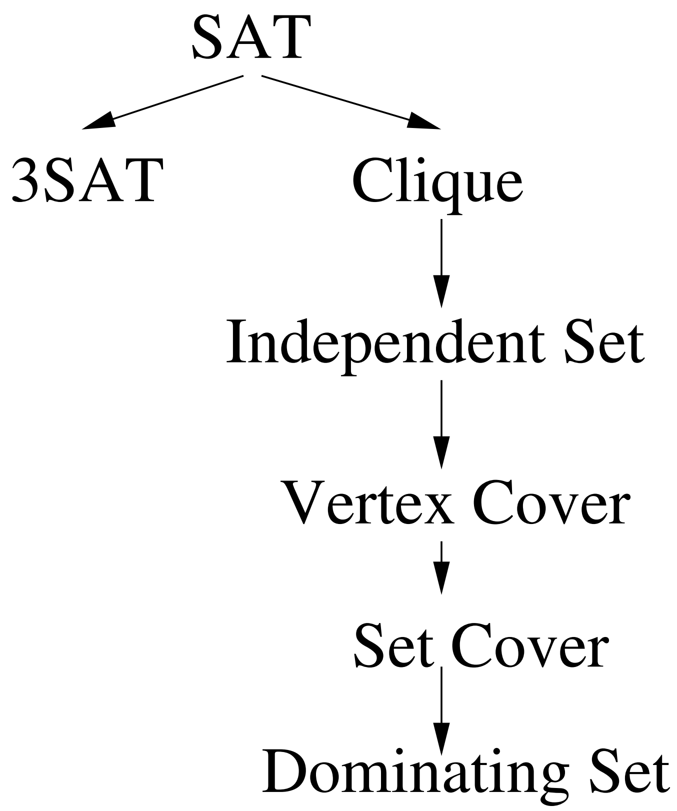


Claim: There is a set cover of size $\leq L$ if and only if there is a dominating set of size $\leq L + 1$.

Proof

- Set cover $I \to$ dominating set S:
  - Take all vertices corresponding to sets in I
  - Take vertex u
  - We get a dominating set S of size $|I|+1$
- Dominating set $S \to$ set cover I
  - Transform the dominating set so that
    * Vertex u is in S. To this end, observe that either u or v must be in S. If v is in S and u is not, we remove v and add u. The result is still a dominating set.
    * S does not contain any u_i. To this end, for any $u_i \in S$ we
      · Remove u_i from S
      · Add any vertex S_j with an edge to u_i

      The resulting set is still a dominating set.
  - The set cover I is defined by nodes $S_i \in S$.


SAT
3SAT
Clique
Independent Set
Vertex Cover
Set Cover
Dominating Set