Lecture 4: February 14, 2002

Today:

- *Randomized* algorithm for sorting: Quicksort

Randomized algorithms

- Can make decisions based on coin flips
- Helps to "fool" adversary
- This is <u>not</u> average case !

More on randomized algorithms

- We can choose an element x uniformly at random from a specified range
- The time could be a random variable $\Rightarrow$ need to bound
  - the *expected* time
  - the probability the time exceeds certain value
- The correctness could be a random variable as well (does not include homeworks)

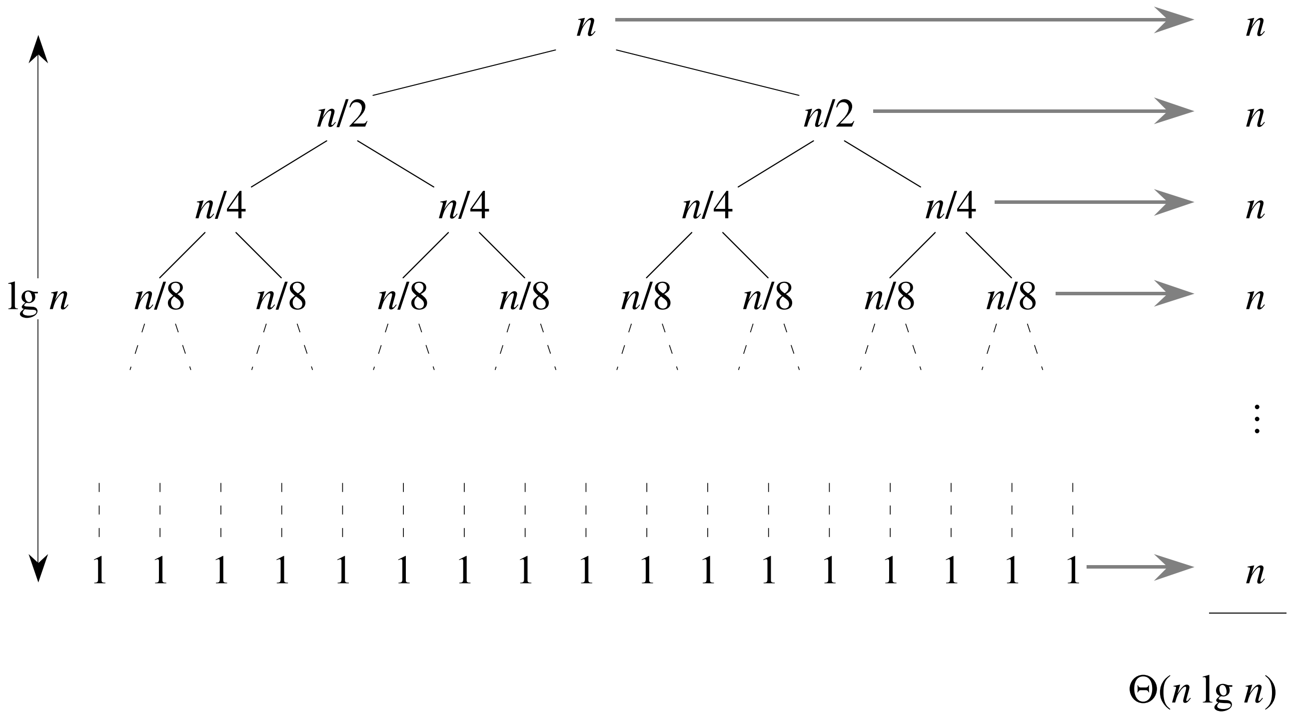
Quicksort

Yet another sorting algorithm, but:

- Sorts "in place" (doesn't require additional array)
  - like insertion sort
  - unlike merge sort
- Very practical (with tuning)
- $\Theta(n \log n)$ time with "high" probability

Merge sort

1. **Divide:** Partition array into 2 subarrays of equal size.
2. **Conquer:** Recursively sort subarrays.
3. **Combine:** Merge two sorted subarrays.


n
n/2
n/2
n/4
n/4
n/4
n/4
lg n
n/8
n/8
n/8
n/8
n/8
n/8
n/8
n/8
1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
n
n
n
n
n
Θ(n lg n)

Quicksort

1. **Divide:** Partition array into 2 subarrays such that elements in lower part $\leq$ elements in higher part.

$\leq x$	$\geq x$

2. **Conquer:** Recursively sort 2 subarrays.
3. **Combine:** Trivial (because in place).

PARTITION procedure

PARTITION(A, p, r)

(Partition $A[p..r]$ around random element $x = A[k]$)

$k \leftarrow \text{Random}(p \ldots r)$

$x \leftarrow A[k]$

$i \leftarrow p - 1$

$j \leftarrow r + 1$

while TRUE **do**

  repeat $j \leftarrow j - 1$ **until** $A[j] \leq x$

  repeat $i \leftarrow i + 1$ **until** $A[i] \geq x$

  if $i < j$ **then** exchange $A[i] \leftrightarrow A[j]$

  else quit (and return j)

Correctness proof idea

Loop invariant:

$$
\begin{array}{lllll}
\hline
| \quad \leq x & | & \quad ? & | & \geq x \\
\hline
p & i & & j & \qquad r
\end{array}
$$

Time $= \Theta(n)$ for n-element subarray

Quicksort — Recursive algorithm

$\textsc{Quicksort}(A, p, r)$
  if $p < r$
    then $q \leftarrow \textsc{Partition}(A, p, r)$
      $\textsc{Quicksort}(A, p, q - 1)$
      $\textsc{Quicksort}(A, q + 1, r)$

Initial call: $\textsc{Quicksort}(A, 1, length[A])$

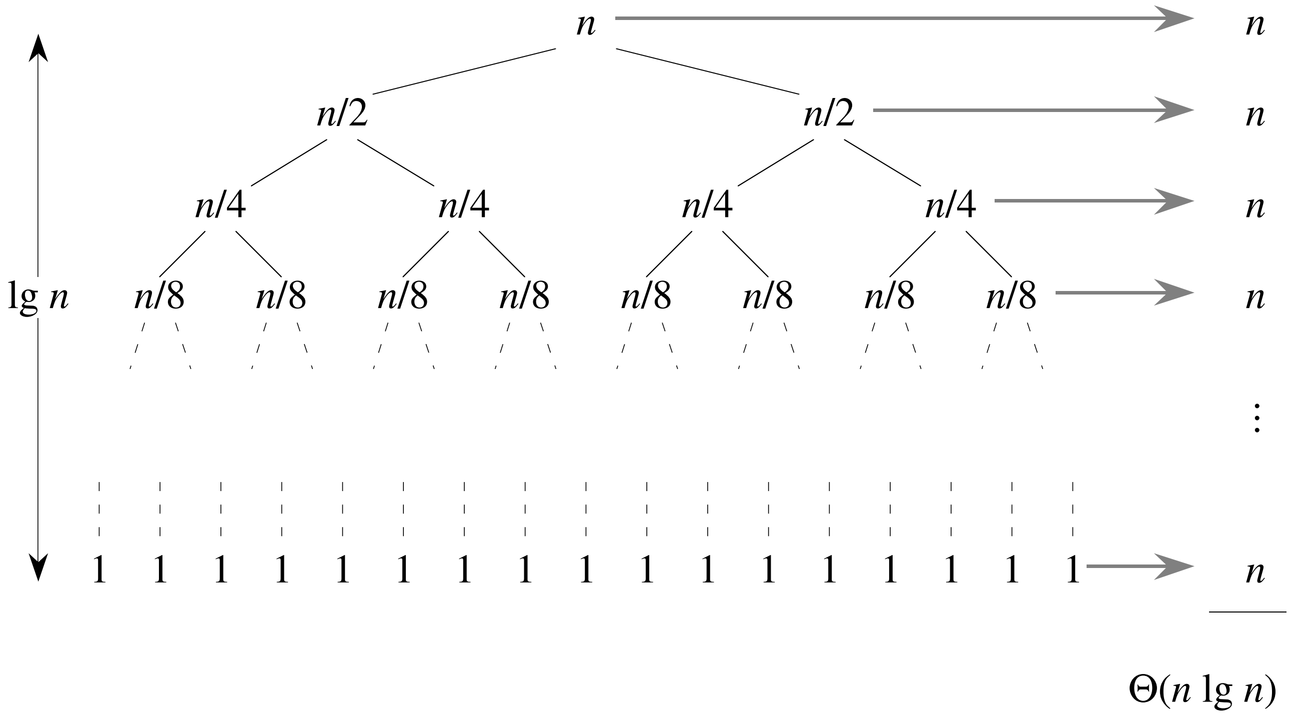
Quicksort analysis

- Correctness - done
- Running time ?

Analysis of QUICKSORT (best case)

If we're lucky, PARTITION always splits array evenly

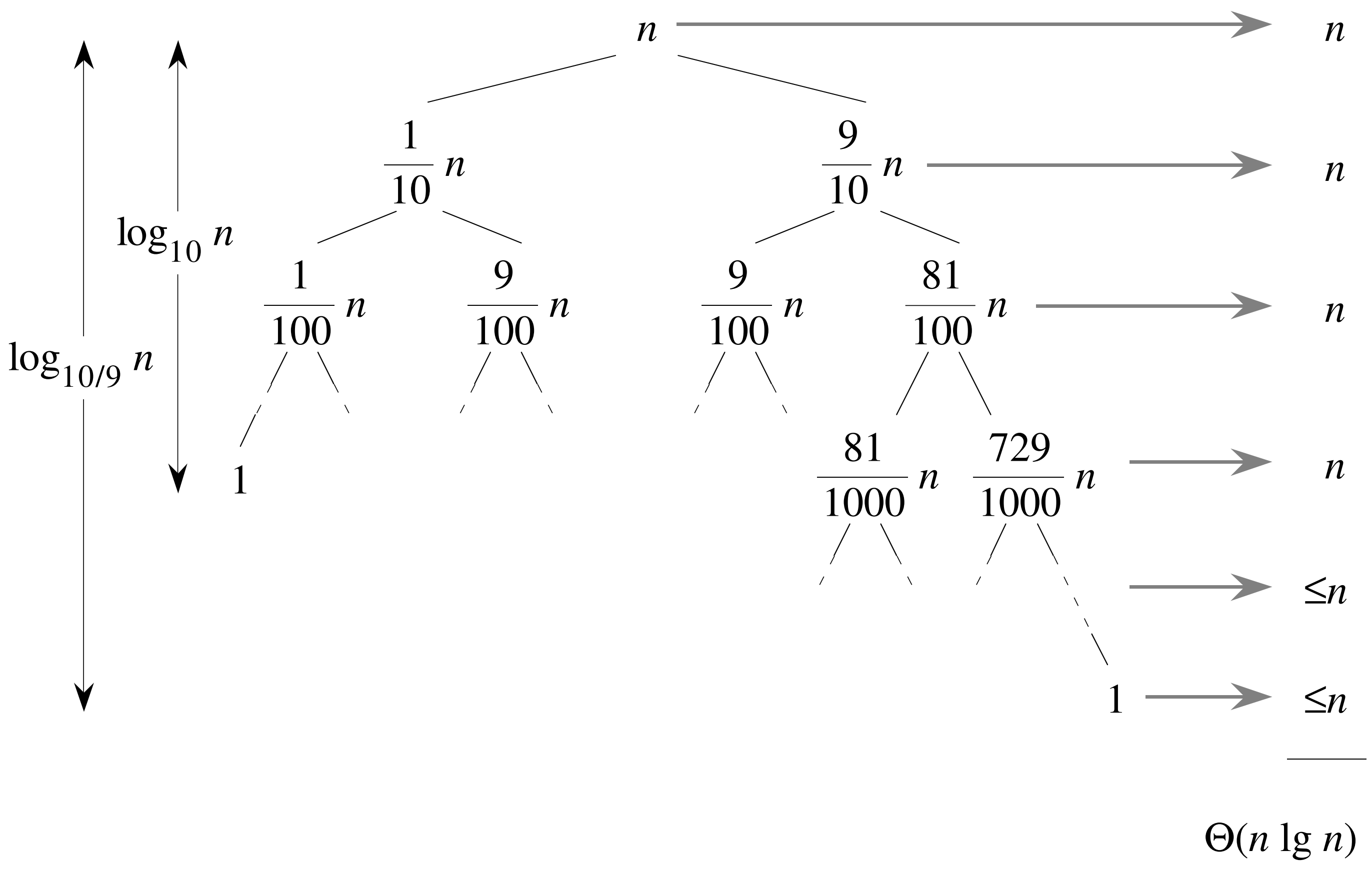
$$T(n) = 2T(n/2) + \Theta(n)$$



Analysis of QUICKSORT (OK case)

Suppose the split is $\frac{1}{10} : \frac{9}{10}$

$$\begin{aligned} T(n) &= T(n/10) + T(9n/10) + \Theta(n) \\ &= \Theta(n \lg n) \quad \text{Still lucky!} \end{aligned}$$



Analysis of QUICKSORT (worst case)

How might we be unlucky?

- one side of partition has 1 element

$$\begin{aligned} T(n) &= T(1) + T(n-1) + \Theta(n) \\ &= T(n-1) + \Theta(n) \qquad \text{because } T(1) = \Theta(1) \\ &= \sum_{k=1}^{n} \Theta(k) = \Theta\Big(\sum_{k=1}^{n} k\Big) \\ &= \Theta(n^2) \quad \text{[arithmetic series]} \end{aligned}$$

Analysis of QUICKSORT

- Optimistic case
- Semi-optimistic case
- Worst case
- ...
- Need more scientific approach
- Will show the time is at most $Cn \log n$ with *high probability*

Key ideas

- The running time bounded by $O(n)$ times the depth of the recursion tree (as seen on earlier pictures)
- "Nice splits" happen with "nice probability"
- If we have large enough number of trials, and each trial has "nice probability" of success, we have many successes with high probability
- For technical reasons, assume all input elements are distinct. Picking random index of element is equivalent to picking random element.

Lucky partitions

Let a_i be the i-th smallest element in $A[\cdot]$ (i.e., with the rank i).

Let D_i denote the depth of a_i in the recursion tree. The total tree depth is $D = \max_i D_i$.

We say a split (say, of $A[p \ldots r]$) is "lucky" for a_i, if the part where a_i ends up in is of size $\leq 3/4(r - p + 1)$, i.e., if we reduce the array size by a factor of $3/4$.

a_i	x	

After $\log_{4/3} n$ lucky splits, a_i is a leaf !

“Lucky” lemma

Lemma: At any time, a_i is lucky with probability at least 1/2.

Proof: To get a lucky split, it is sufficient that x has rank in $\{n/4 \ldots 3n/4\}$. This happens with probability $\geq 1/2$.

The analysis

Let $l = C \log_{4/3} n$, where C is a "large" constant. What is the probability that $D_i > l$?

It is at most the probability that in l trials, each having success probability $\geq 1/2$, we were *not* successful $\geq l - \log_{4/3} n$ times.

The latter probability is <u>at most</u>

$$\binom{l}{l-t} 1/2^{l-t}$$

for $t = \log_{4/3} n$.

The analysis ctd.

$$\begin{aligned}
\binom{l}{l-t} 1/2^{l-t} &= \binom{l}{t} 1/2^{l-t} \\
&\le \left(\frac{el}{t}\right)^t 1/2^{l-t} \\
&= \left(\frac{eC\log_{4/3} n}{\log_{4/3} n}\right)^t 1/2^{(C-1)\log_{4/3} n} \\
&= \left(\frac{eC}{2^{C-1}}\right)^{\log_{4/3} n}
\end{aligned}$$

When C is large enough (e.g., 20), the probability is smaller than $1/n^2$.

Finishing the analysis

We proved that $\Pr[D_i > l] \leq 1/n^2$.

Therefore, the probability that *there exists* i such that $D_i > l$ is at most

$$\Pr[D_1 > l] + \Pr[D_2 > l] \ldots \Pr[D_n > l] \leq n \cdot 1/n^2 = 1/n$$

Therefore probability that $\max_i D_i \leq l$ is at least $1 - 1/n$.

Appendix

Why is the probability that "in l trials, each having success probability $\geq 1/2$, we were *not* successful $\geq l-t$ times" at most

$$\binom{l}{l-t} 1/2^{l-t}$$

?

- if we were not successful $\geq l-t$ times, we were not successful during some set of $l-t$ trials
- the probability we were not successful during a *fixed* set of exactly $l-t$ trials is at most $1/2^{l-t}$
- there are $\binom{l}{l-t}$ subsets of trials of size $l-t$