6.045J/18.400J: Automata, Computability and Complexity Prof. Nancy Lynch

Recitation 2: DFAs and NFAs – Solutions to some problems

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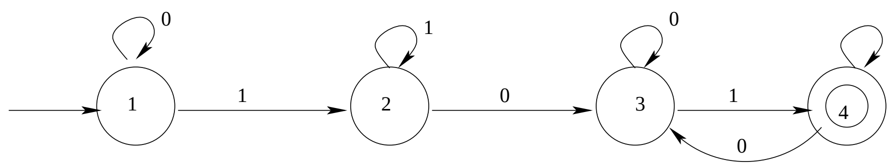
Problem 1: What language L does the following automaton recognize ? Prove that the automaton indeed recognizes the language you think it recognizes. To do this, we will need to prove that our FA (1) accepts all strings in L and (2) does not accept any string not in L.



1. Forward direction (accepts all strings in L).
2. Reverse direction (does not accept any string outside of L).

Solution 1: The machine recognizes the language $L = \{x \mid x \text{ contains a 10 and ends in } 1\}$.

1. Suppose x is in L. Then x is of the form $y1$ for some y, such that y has 10 as a substring (Intuitively, here we disconnect the two conditions that lead to acceptance. Note that it might not always be possible to do this).

   Now, we prove that $\delta^*(1, y) \in \{3, 4\}$. This suffices to prove our claim, since from any of the states in $\{3, 4\}$, if the machine reads a 1, it goes to the accept state (and therefore, the string $x = y1$ is accepted). Consider the first occurence of the substring 10 in y. Write y as $y_1 10 y_2$, where y_1 does not contain any 10. (Intuitively, y_1 is either the string of all 0s or a string formed by some 0s followed by some 1s). We prove by induction on the length of y_1 that **if y_1 does not contain a 10, then** $\delta^*(1, y_1) \in \{1, 2\}$. This is certainly true for an empty string. The induction hypothesis is that,

   - If a string z ends in a 0, $\delta^*(1, z) = 1$
   - If a string z ends in a 1, $\delta^*(1, z) = 2$.

   Assume the induction hypothesis for a string z length k.

   - Assume y_1 ends in 0. Therefore, $y_1 = z0$ for some z of length k. Then, since y_1 does not contain a 10, z must end in a 0. Therefore, $\delta^*(1, y_1) = \delta(\delta^*(1, z), 0) = \delta(1, 0) = 1$.
   - Now, assume $y_1 = z1$. Now, z could end in a 0 or a 1. Say z ended in a 0. Then $\delta^*(1, y_1) = \delta(\delta^*(1, z), 1) = \delta(1, 1) = 2$ (Because $\delta^*(1, z) = 1$ if z ends in 0). Say z ended in a 1. Then $\delta^*(1, y_1) = \delta(\delta^*(1, z), 1) = \delta(2, 1) = 2$. Thus, the induction hypothesis is true for y_1 too.

   Now, note that from any of the states $\{1, 2\}$, after reading a 10, the machine lands up in state 3. Therefore, $\delta^*(1, y_1 10) = 3$. After reaching one of the states in $\{3, 4\}$, the machine cannot go back to any of the states in $\{1, 2\}$. Therefore, $\delta^*(1, y_1 10 y_2) \in \{3, 4\}$. Furthermore, $\delta^*(1, x) = \delta^*(1, y_1 10 y_2 1) = \delta(\{3, 4\}, 1) = 4$. (**Notational Clarification :** By $\delta(S, a)$ for a set $S \subseteq Q$, we mean $\bigcup_{q \in S} \delta(q, a)$).

2. Now suppose M accepts a string x not in L. Either x does not end in a 1 or it does not contain a substring 10. Clearly, if x does not end in a 1 it does not reach the accept state, state 4. On the other hand, suppose it does not contain 10 as a substring. (*Go back to the previous proof and reuse parts of it.* How ?) Remember, we showed that if a string z does not contain a 10, it ends in one of the states $\{1, 2\}$. We are done. ■

Problem 2: **Optional** (if we have enough time)
An *all-paths-NFA* M is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ that accepts $x \in \Sigma^*$ if and only if *every* possible state that M could be in after reading x is a state from F. Prove that all-NFAs recognize exactly the regular languages. (Notice the contrast with NFAs)

Solution 2: Note, first, that an all-paths-NFA has the same syntax as an NFA, *but their acceptance criteria are different*. An NFA accepts a string x if and only if *at least one* of the computation paths leads to an accepting state. On the other hand, an all-paths-NFA accepts a string if and only if *every* computation path ends in an accepting state. In particular, the same machine can act as an NFA and an all-paths-NFA, depending on the acceptance criterion it uses.

Now, on to the solution: M is an all-paths-NFA. The gameplan is to show that, for every L that is accepted by some all-paths-NFA M, there is an NFA N that accepts $\bar{L}$. This will prove that, if L is accepted by some all-paths-NFA, $\bar{L}$ is regular. Since the class of regular languages is closed under complement, L is regular too.

Let $L = L(M)$. We show that the NFA $N = (Q, \Sigma, \delta, q_0, F' = Q - F)$, accepts $\bar{L}$. Suppose $x \in L$. Then, by definition, all the computation paths of M on x leads to a state in F. Which means none of the computation paths end in $F' = Q - F$. Therefore, N (an NFA) does not accept x, and $x \notin L(N)$. It is not hard to see that, if $x \notin L$ (that is, it is not accepted by M), then x is accepted by N, and therefore $x \in L(N)$. (We leave this as an exercise). We have shown that, $x \in L(M)$ if and only if $x \notin L(N)$. Therefore, N accepts $\bar{L}$. ■