Recitation 1: Discussion Material

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Proof Techniques

Problem 1: **Proof by Contradiction**: If there are 6 people at a party shaking hands, then there must be at least two people who shook hands with the same number of other people.

Solution 1: Some people in the recitation suggested (in essence) the following alternate solution: Construct a graph G whose nodes are the 6 people, and node x is connected to node y if x shakes hand with y. If we assume the negation of the statement to be proved (as is usual in proofs by contradiction), then we have that, every person shook hands with a different number of other people. Thus, there are people who shook hands with $0, 1, 2, 3, 4$ and 5 other people. In graph-theoretic terms, this means that there are vertices in the graph with all possible degrees $d \in \{0, 1, 2, \ldots, 5\}$. Now, count the total degree of all nodes in the graph: this is $0+1+\ldots+5 = 15$, which is odd. But the total degree of all nodes in *any* graph *has to be* even. (See chapter 0 of Sipser's book if you need a proof of this fact). Behold ! We have the contradiction.

One shortcoming of this approach is that, we cannot, for instance, prove that "If there are 8 people at a party shaking hands, then there must be at least two people who shook hands with the same number of other people.", which is a true statement.

The proof presented in the recitation went as follows: Assume that everybody shook hands with different number of people (the negation of what we want to prove). Then, (as before) there are people who shook hands with $0, 1, 2, 3, 4$ and 5 other people. Lets say Alice shook hands with *nobody else* and Bob shook hands with all the other people. This is clearly a contradiction, since Bob could not have shaken hands with Alice. This argument generalizes to prove that "If there are k people at a party shaking hands, then there must be at least two people who shook hands with the same number of other people.", for any value of $k > 1$.

A wider perspective: For a graph G on n nodes, define the degree sequence of G to be the sequence of the degrees of all the n nodes (sorted in the ascending order, just so that it is unique). The problem we just solved is a special case of the more general problem of determining, for a given degree sequence $(d_1, d_2, \ldots, d_n)$, whether there exists a graph G on n nodes that has the given degree sequence.

Problem 2: **Proof by Induction (Base Case) (Induction Hypothesis) (Inductive Step)**: Now *correctly* prove the following statement: $\forall n \in \mathbb{N},\ n^3 - n$ is divisible by 6.

Solution 2: In the first section of the recitation (at 10am), we did only the proof of contradiction of this statement. Here is the proof by induction.

1. **Basis Step** : 6 divides $1^3 - 1 = 0$, clearly.

2. **Induction Step** : Suppose $6|k^3 - k$. We want to prove that $6|(k+1)^3 - (k+1)$ too. Note that $(k+1)^3 - (k+1) = k^3 + 3k^2 + 2k = (k^3 - k) + 3k(k+1)$. Now, $6|k^3 - k$, by assumption. $6|3k(k+1)$ too, since $3|3k(k+1)$ (trivially) and $2|k(k+1)$ (which is a product of two consecutive integers). Thus $6|(k^3 - k) + 3k(k+1) = (k+1)^3 - (k+1)$.

Problem 3: **Double Induction**:

Let the function $R(s,t)$ (for $s, t \in \mathbb{N}$) be defined by the induction:

$$R(s,t) = R(s,t-1) + R(s-1,t)$$

and the base cases

$$R(s,2) = R(2,s) = s \quad (\text{ for all } s \in \mathbb{N})$$

Prove that $R(s,t) \leq \binom{s+t-2}{s-1}$.

Solution 3: In the 1pm recitation, we could not quite finish the solution to this problem. Here it is:

The basis case is all (s,t) in which either $s = 2$ or $t = 2$. $R(s,2) = s = \binom{s+2-2}{s-1}$. Also, $R(2,s) = s = \binom{s+2-2}{2-1}$.

For the inductive hypothesis, assume that the statement is true for all (s',t') such that $s'+t' < s+t$. (Alternatively, we could assume that the statement is true for all (s',t') such that either $s' < s$ or $t' < t$).

Now, $R(s,t) = R(s,t-1) + R(s-1,t)$ (by definition). By the inductive hypothesis, we know that

$$R(s,t-1) \leq \binom{s+t-3}{s-1} and$$

$$R(s-1,t) \leq \binom{s+t-3}{s-2}$$

$R(s,t) = R(s,t-1)+R(s-1,t) \leq \binom{s+t-3}{s-1}+\binom{s+t-3}{s-2} = \frac{(s+t-3)!}{(s-1)!(t-2)!}+\frac{(s+t-3)!}{(s-2)!(t-1)!} = \frac{(s+t-3)!}{(s-2)!(t-2)!}(\frac{1}{s-1}+\frac{1}{t-1}) = \frac{(s+t-2)!}{(s-1)!(t-1)!} = \binom{s+t-2}{s-1}$. QED.

The non-trivial step is to find the right inductive hypothesis.