DISCRETE MATHEMATICS 8 (1974) 31–40. © North-Holland Publishing Company

ASYMPTOTIC BOUNDS FOR THE NUMBER OF CONVEX n-OMINOES*

David A. KLARNER and Ronald L. RIVEST

Computer Science Department, Stanford University, Stanford, Calif. 94305, USA

Received 18 May 1973**

Abstract. Unit squares having their vertices at integer points in the Cartesian plane are called *cells.* A point set equal to a union of n distinct cells which is connected and has no finite cut set is called an *n-omino.* Two n-ominoes are considered the same if one is mapped onto the other by some translation of the plane. An n-omino is *convex* if all cells in a row or column form a connected strip. Letting $c(n)$ denote the number of different convex n-ominoes, we show that the sequence $((c(n))^{1/n}: n = 1, 2, ...)$ tends to a limit γ, and $\gamma = 2.309138...$.

1. Introduction

Unit squares having their vertices at integer points in the Cartesian plane are called *cells.* A point set equal to a union of n distinct cells which is connected and has no finite cut set is called an *n-omino.* Two n-ominoes are considered the same if one is mapped onto the other by some translation of the plane. (Such n-ominoes were called *fixed* animals with n cells by Read [8].) For example, there are six different 3-ominoes as shown in Fig. 1.

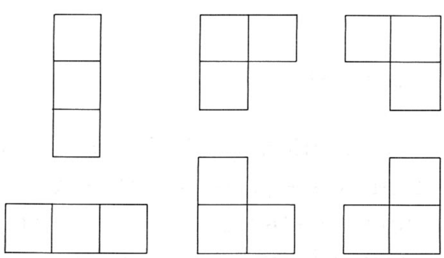

Fig. 1. The 3-ominoes.

* This research was supported in part by National Science Foundation Grant GJ-992, and Office of Naval Research Contract N-00014-67-A-0112-0057 NR 044-402.

** Original version received 29 December 1972.

Let $t(n)$ denote the number of distinct n-ominoes. It is known [3] that the sequence $((t(n))^{1/n} : n = 1, 2, ...)$ tends to a limit θ. The investigation of θ began with Eden's work [1]; he managed to prove that $3.14 < \theta \leq 6.75$. There has been considerable effort expended to improve these bounds. Currently, the best lower bound (given in [3]) is $3.72 < \theta$, while the best upper bound (given in [5]) is $\theta < 4.65$.

An n-omino is *row-convex* when each row of the n-omino is a connected strip of cells. *Column-convex* n-ominoes are defined analogously. All six of the 3-ominoes (shown in Fig. 1) are both row-convex and column-convex; in general, such n-ominoes are said to be *row-column-convex,* or just *convex* for short. It was shown in [2] (and in [3] by a second method) that

$$(1) \qquad \frac{x\,(1-x)^3}{1-4x+7x^2-5x^3} = \sum_{n=1}^{\infty} b(n)\,x^n \,,$$

where $b(n)$ denotes the number of distinct row-convex n-ominoes. (This result was also obtained by Pólya [6].) Thus it follows that the sequence $((b(n))^{1/n} : n = 1, 2, ...)$ tends to a limit β which is equal to the largest real root of $y^3 - 4y^2 + 7y - 5 = 0$; that is, $\beta = 3.20...$.

Recently, Donald Knuth wrote us and asked us if the number $c(n)$ of convex n-ominoes had been investigated. This paper is entirely motivated by Knuth's question. We shall be concerned with the problem of effectively calculating the limit γ of the sequence $((c(n))^{1/n} : n = 1, 2, ...)$. One of the first things we prove is that this limit exists. Later on we show how to calculate upper and lower bounds for γ and give the best results obtained by these methods.

2. Existence of $\lim_{n\to\infty} (c(n))^{1/n}$

Following Ceasar's admonition, we divide, then conquer. A convex n-omino may be split into three parts by making two cuts between certain rows so that the upper and lower parts are roughly trapezoids and the middle part is roughly a parallelogram. A typical sectioning of this sort is shown in Fig. 2. More precisely, the trisection of a convex n-omino A is accomplished by cutting along the lowest level of A where the left boundary of A goes to the right and by cutting along the lowest level of A where the right boundary of A goes to the left.

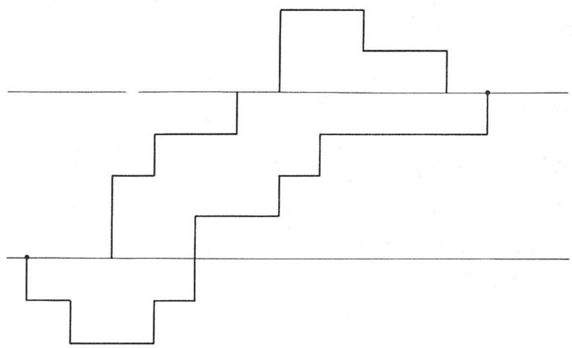

Fig. 2. Trisection of a convex 28-omino..

A convex n-omino whose left boundary climbs to the right and whose right boundary climbs to the left corresponds to a partition of n called a *stack* by Wright [9]. We let $s(n)$ denote the number of distinct n-ominoes corresponding to stacks; for example, there are four 3-ominoes shown in Fig. 1 which correspond to stacks, so $s(3) = 4$. A convex n-omino whose left and right boundaries both climb to the right is called a *parallelogram,* and $p(n)$ will denote the number of distinct n-ominoes which are parallelograms. Clearly, $p(n) \leq c(n)$ for all n; also, $s(n) \leq p(n)$ for all n (the diagram in Fig. 3 suggests a proof of this fact). Finally, an obvious construction establishes that $p(m)\,p(n) \leq p(m+n)$ for all m, n. Now we use the fact that if $\{u_n\}$ is a sequence of natural numbers such that $((u_n)^{1/n}: n = 1, 2, ...)$ is bounded and $u_m\, u_n \leq u_{m+n}$ for all m, n, then $\lim_{n\to\infty} (u_n)^{1/n}$ exists. (For similar results, see Pólya and Szegö

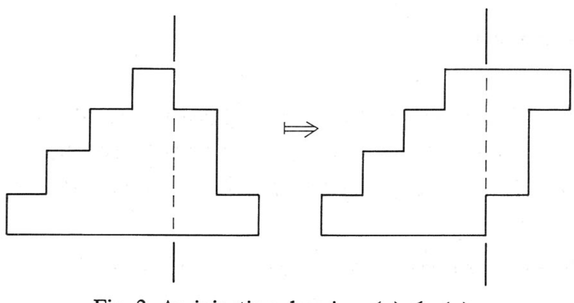

Fig. 3. An injection showing $s(n) \leq p(n)$.

[7, p. 171].) We have $p(n) \leq b(n) < (3.20)^n$ for all large n, and $p(m)\,p(n) \leq p(m+n)$, so

$$\lim_{n\to\infty} (p(n))^{1/n} = \gamma \tag{2}$$

exists. Using the fact that every convex n-omino splits into two stacks and one parallelogram, we can reconstruct these n-ominoes by pasting together two stacks and one parallelogram in various ways. Again, using an obvious construction, and using the fact that $p(i)\,p(j)\,p(k) \leq p(i+j+k)$ for all i, j, k, it is easy to show that

$$c(n) \leq 2n^2 \sum_{(i,j,k)} s(i)\,p(j)\,s(k) \leq 2n^2 \sum_{(i,j,k)} p(i)\,p(j)\,p(k) \tag{3}$$

$$\leq 2n^2 \binom{n+2}{2} p(n) \leq (n+2)^4\, p(n)\,,$$

where the index of summation in the sums extends over all compositions (i, j, k) of n into non-negative parts. There are $\binom{n+2}{2}$ such compositions.

Using (2) and (3) together with the fact that $p(n) \leq c(n)$ for all n, we have

$$\gamma = \lim_{n\to\infty} (p(n))^{1/n} \leq \liminf_{n\to\infty} (c(n))^{1/n} \tag{4}$$

$$\leq \limsup_{n\to\infty} (c(n))^{1/n} \leq \lim_{n\to\infty} ((n+2)^4\, p(n))^{1/n} = \gamma\,.$$

Hence $\lim_{n\to\infty} (c(n))^{1/n}$ exists, and

$$\lim_{n\to\infty} (c(n))^{1/n} = \lim_{n\to\infty} (p(n))^{1/n} = \gamma\,. \tag{5}$$

3. An integral equation

We shall use a theory developed in [4] concerning a double sequence $(b(n, a) : n, a = 1, 2, \ldots)$ defined in terms of given sequences $(f(m, n) : m, n = 1, 2, \ldots)$ and $(g(n) : n = 1, 2, \ldots)$ as follows:

$$b(n, a) = \sum f(a_1, a_2)\,f(a_2, a_3) \ldots f(a_{k-1}, a_k)\,g(a_k)\,, \tag{6}$$

where the index of summation extends over all k-tuples $(a_1, \ldots, a_k)$ of natural numbers for $k = 1, \ldots, n$ with $a_1 = a$ and $a_1 + \ldots + a_k = n$. It was shown that if

$$G(x) = \sum_{n=1}^{\infty} g(n)\, x^n \tag{7}$$

and

$$F(x, y) = \sum_{m,n=1}^{\infty} f(m, n)\, x^m\, y^n \tag{8}$$

converge for $|x|$ and $|y|$ sufficiently small, then

$$B(x, y) = \sum_{n=1}^{\infty} \sum_{a=1}^{n} b(n, a)\, y^a\, x^n \tag{9}$$

converges for $|x|$ and $|y|$ sufficiently small, and

$$B(x, y) = G(xy) + \frac{1}{2\pi i} \int_C F(xy,\ 1/s)\, B(x, s) \frac{ds}{s}\ , \tag{10}$$

where C is a contour in the s-plane which includes $s = 0$ and the singularities of $F(xy,\ 1/s)$ but excludes the singularities of $B(x, s)$. The theory of (10) runs parallel to that of the Fredholm integral equation. In particular, if $F(x, y)$ has the special form

$$F(x, y) = R_1(x)\, S_1(y) + \ldots + R_t(x)\, S_t(y)\ , \tag{11}$$

we say F is *separable,* and it turns out that (10) can be converted into a system of t equations linear in t unknown functions. The system can be solved and the solution yields a formula for $B(x, y)$. We shall give an example of this later on.

If F is not separable, we can still get information about B by approximating F with something that is separable. Suppose

$$K(x, y) = \sum k(m, n)\, x^m\, y^n \tag{12}$$

and $k(m, n) \leq f(m, n)$ for all m, n, then we say K is *a lower bound* on F, an *upper bound* on F is defined analogously. If K is separable, we may substitute K for F in (10) and calculate a lower bound for B. Upper bounds for B may be obtained in a similar fashion. We shall adopt this strategy too, so an example is forthcoming.

The relevance of the foregoing discussion to the enumeration of n-celled parallelograms is as follows: the number of $(m+n)$-celled parallelograms having m cells in one row and n cells in a second row is

(13) $$f(m, n) = \min\{m, n\}\ .$$

It is fairly easy to show that the number of n-celled parallelograms with exactly k rows of cells having exactly a_i cells in the ith row for $i = 1, \ldots, k$ is

(14) $$f(a_1, a_2) f(a_2, a_3) \ldots f(a_{k-1}, a_k)\ .$$

Thus, if we take f as defined in (13) and put $g(j) = 1$ for all j, we can sum (6) over $a = 1, \ldots, n$ and obtain $p(n)$. In this case, we have

(15) $$F(x, y) = xy/(1-x)(1-y)(1-xy)\ ,$$

(16) $$G(x) = x/(1-x)\ .$$

Substituting these functions in (10) gives

(17) $$B(x, y) = \frac{xy}{1-xy} + \frac{1}{2\pi i} \int_C \frac{xy\, B(x, s)\, \mathrm{d}s}{(1-xy)(s-1)(s-xy)}$$

$$= \frac{xy}{1-xy} + \frac{xy}{(1-xy)^2} B(x, 1) - \frac{xy}{(1-xy)^2} B(x, xy)\ .$$

We can iterate (17) to eliminate $B(x, xy)$, $B(x, x^2y)$, ... successively to find

(18) $$B(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}\, x^{k(k+1)/2}\, y^k (1-x^k y + B(x, 1))}{(1-xy)^2\, (1-x^2y)^2 \ldots (1-x^k y)^2}\ .$$

Setting $y = 1$ in (18), we solve for $B(x, 1)$, the generating function of $(p(n)\colon n = 1, 2, \ldots)$, which turns out to be

$$(19)\quad B(x,1)=\frac{\dfrac{x}{1-x}-\dfrac{x^3}{(1-x)^2\,(1-x^2)}+\dfrac{x^6}{(1-x)^2\,(1-x^2)^2\,(1-x^3)}-\dots}{1-\dfrac{x}{(1-x)^2}+\dfrac{x^3}{(1-x)^2(1-x^2)^2}-\dfrac{x^6}{(1-x)^2(1-x^2)^2(1-x^3)^2}+\dots}$$

$$=\sum_{n=1}^{\infty} p(n)\,x^n\ .$$

We have been unable to make use of (19) in estimating $p(n)$. Instead, we use upper and lower bounds for F as defined in (15), and then use (10) to calculate upper and lower bounds for B.

4. Lower bounds

Let

$$(20)\qquad F_k(x,y)=\sum_{m,n=1}^{k} f(m,n)\,x^m\,y^n\ ,$$

where $f(m, n) = \min\{m, n)$ just as in (13), and let $B_k(x, y)$ denote the solution of (10) having F_k substituted for F. Since F_k is a lower bound for F, it follows that B_k is a lower bound for B. It was shown in [4] that when the kernel of (10) is approximated by a polynomial as in this case, then $B_k(x, 1)$ is a rational function, say $B_k = P_k/Q_k$ with P_k and Q_k polynomials, and the denominator of B_k may be expressed as a determinant. In the present situation this turns out to be

$$(21)\qquad Q_k(x)=\begin{vmatrix} 1-x & 1 & 1 & \dots & 1\\ 1 & 2-x^2 & 2 & \dots & 2\\ 1 & 2 & 3-x^3 & \dots & 3\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & 2 & 3 & \dots & k-x^k\end{vmatrix}.$$

If we put $Q_0(x) = 1$ and $Q_1(x) = 1-x$, we can use (21) to verify that

$$(22)\qquad Q_k(x)=(1-x^{k-1}-x^k)\,Q_{k-1}(x)-x^{2k-2}\,Q_{k-2}(x)$$

for $k = 2, 3, \ldots$. For example,

$$Q_2(x) = 1-2x-x^2+x^3 \,,$$
$$Q_3(x) = 1-2x-2x^2+2x^3+2x^4+x^5-x^6 \,,$$
$$Q_4(x) = 1-2x-2x^2+x^3+3x^4+5x^5-2x^6-2x^7-2x^8-x^9+x^{10} .$$

Letting γ_k denote the largest real root of $Q_k(1/x) = 0$, we have $\gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma$, where γ is defined in (2). We have used a computer to calculate lower bounds for $\gamma_1, \gamma_2, \ldots, \gamma_{10}$ given in Table 1. Our results indicate that the sequence $\{\gamma_i\}$ converges very quickly to the value 2.30913859..., our best lower bound for γ.

Table 1

k	γ_k	β_k
1	1.00000000	2.41421356
2	2.24697960	2.33578290
3	2.30855218	2.31475605
4	2.30913772	2.31023504
5	2.30913859	2.30934711
6	2.30913859	2.30917790
7	2.30913859	2.30914598
8	2.30913859	2.30913998
9	2.30913859	2.30913885
10	2.30913859	2.30913864

5. Upper bounds

For $k = 1, 2, \ldots$, we define upper bounds $f^k(m, n)$ for $f(m,n) = \min\{m, n\}$ as follows:

$$(25) \qquad f^k(m, n) = \begin{cases} m & \text{if } k < n < m, \\ f(m, n) & \text{otherwise.} \end{cases}$$

Hence

$$(24) \qquad F^k(m, n) = \sum_{m,n=1}^{\infty} f^k(m, n)\, x^m y^n$$
$$= \frac{xy}{(1-x)^2(1-y)} - \frac{x^2 y}{(1-x)^2} - \ldots - \frac{x^{k+1} y^k}{(1-x)^2}$$

is an upper bound for F; furthermore, note that F^k is separable.

Let B^k denote the solution of (10) with F^k substituted for F. Then

$$B^k(x,y) = \frac{xy}{1-xy} + \frac{xyB^k(x,1)}{(1-xy)^2} - \frac{xy}{(1-xy)^2} \sum_{r=1}^{k} x^r y^r B_r^k(x)\,, \tag{25}$$

where

$$B_r^k(x) = \frac{1}{k!} \frac{\partial^r}{\partial s^r} B^k(x,s) \Big|_{s=0} \,.$$

Now we use (25) to get a system of equations involving $B_1^k, \ldots, B_k^k$. Take the rth partial derivative with respect to y at $y = 0$ and divide by $r!$ in (25) to get

$$B_r^k(x) = x^r + r\,x^r B^k(x,1) - \sum_{j=1}^{r-1} (r-j)\,x^r B_j^k(x)\,, \tag{26}$$

from which it follows that

$$B_{r+1}^k(x) = (2x - x^{r+1})\,B_r^k(x) - x^2 B_{r-1}^k(x)\,. \tag{27}$$

Setting $B_r^k(x) = P_r(x) + Q_r(x)B^k(x,1)$ for $r = 1, \ldots, k$, it follows that P_r and Q_r also satisfy the difference equation (27). Also we can substitute $P_r + Q_r B^k$. For B_r in (25) with $y = 1$ and solve for $B^k(x,1)$ in terms of $P_1, Q_1, \ldots, P_k, Q_k$ to obtain

$$B^k(x,1) = \frac{x - x^2 - \Sigma_{j=1}^k x^{j+1} P_j(x)}{1 - 3x + x^2 + \Sigma_{j=1}^k x^{j+1} Q_j(x)}\,. \tag{28}$$

Thus B^k is a rational function whose numerator N_k and denominator D_k we know how to compute because they are defined in terms of $P_1, \ldots, P_k$ and $Q_1, \ldots, Q_k$ which we know how to compute. Let β_k denote the largest real root of $D_k(1/x)$, then we know

$$\lim_{n\to\infty} \left(\sum_{a=1}^{n} b^k(n,a) \right)^{1/n} = \beta_k \le \gamma\,, \tag{29}$$

and $\beta_1 \ge \beta_2 \ge \ldots \ge \gamma$. Thus we can calculate upper bounds for $\beta_1, \beta_2, \ldots$ to obtain successively better upper bounds for γ.

Using the definitions

$$D_k = 1-3x+x^2+x^2 Q_1 + \dots + x^{k+1} Q_k \,, \tag{30}$$

$$Q_{r+1} = (2x-x^{r+1}) Q_r - x^2 Q_{r-1} \qquad (r > 1) \,, \tag{31}$$

and $Q_1 = x$, $Q_2 = 2x^2 - x^3$, the polynomials $D_1, D_2, \dots$ are calculated with relative ease. For example, we found

$$D_1 = 1-3x+x^2+x^3 \,,$$
$$D_2 = 1-3x+x^2+x^3+2x^5-x^6 \,,$$
$$D_3 = 1-3x+x^2+x^3+2x^5-x^6+3x^7-2x^8-2x^9+x^{10} \,.$$

Using a computer, the polynomials $D_1, \dots, D_{10}$ were calculated via (30), and upper bounds for β_k, the largest real root of $D_k(1/x) = 0$, were computed for $1 \leq k \leq 10$ using the Newton–Raphson method. These upper bounds for β_k are given in Table 1.

Combining our upper and lower bounds, we can conlude that

$$\gamma = \lim_{n \to \infty} (c(n))^{1/n} = 2.309138\dots \,. \tag{32}$$

References

[1] M. Eden, A two-dimensional growth process, Proc. Fourth Berkeley Symposium on Mathematical Statistics and Probability, IV (Univ. of California Press, Berkeley, Calif., 1961) 223–239.

[2] D.A. Klarner, Some results concerning polynominoes, Fibonacci Quart. 3 (1965) 9–20.

[3] D.A. Klarner, Cell growth problems, Canad. J. Math. 19 (1967) 851–863.

[4] D.A. Klarner, A combinatorial formula involving the Fredholm integral equation, J. Combin. Theory 5 (1968) 59–74.

[5] D.A. Klarner and R.L. Rivest, A procedure for improving the upper bound for the number of n-ominoes, Canad. J. Math. 15 (3) (1973) 585–602.

[6] G. Pólya, On the number of certain lattice polygons, J. Combin. Theory 6 (1969) 102–105.

[7] G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Vol. 1 (Berlin, 1925).

[8] R.C. Read, Contributions to the cell growth problem, Canad. J. Math. 14 (1962) 1–20.

[9] E.M. Wright, Stacks, Quart. J. Math. Oxford (2) 19 (1968) 313–320.