Today:

- Derandomization via method of Conditional Expectations
- Random Walks!

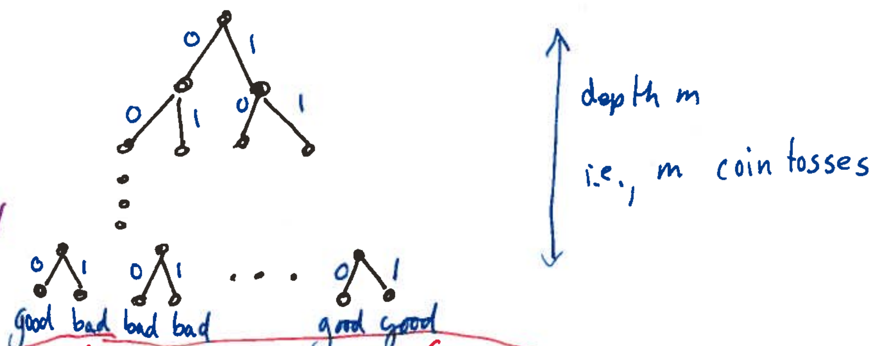
Markov chains + random walks on graphs

Stationary Distributions

More derandomization: The method of conditional expectations

idea: view coin tosses of algorithm as path down a tree of depth m ← # coin tosses

0 = H
1 = T



good = correct/reaches target/good approx/pass...

Alternatively: cut values

$c_{0..0}$ $c_{0..1}$ $c_{0..10}$ $c_{0..11}$ $c_{1....0}$ $c_{1....1}$

good randomized algorithm ⟺ most leaves are good

idea find a good path to leaf "bit-by-bit"

more formally:

Fix randomized algorithm A
input x
m = # random bits used by A on x

for $1 \le i \le m$ + $r_1 \cdots r_i \in \{0,1\}$, let $p(r_1 \cdots r_i)$ = fraction of continuations that end in "good" leaf

$c(r_1 \cdots r_i)$ = ave cut value if set 1st i nodes to $r_1 \cdots r_i$

$$p(r_1 \cdots r_i) = \frac{1}{2} \cdot p(r_1 \cdots r_i\, 0) + \frac{1}{2} \cdot p(r_1 \cdots r_i\, 1)$$

$$= \Pr_{R_{i+1} \cdots R_m}[A(x : r_1 \cdots r_i R_{i+1} \cdots R_m) \text{ correct}]$$

$$= \frac{1}{2}\left[\Pr_{R_{i+2} \cdots R_m}[A(x : r_1 \cdots r_i\, 0\, R_{i+2} \cdots R_m) \text{ correct}]\right.$$

$$+ \frac{1}{2}\left[\Pr_{R_{i+2} \cdots R_m}[A(x : r_1 \cdots r_i\, 1\, R_{i+2} \cdots R_m) \text{ correct}]\right.$$

by averaging, ∃ setting of r_{i+1} to 0 or 1
s.t. $p(r_1 \cdots r_{i+1}) \ge p(r_1 \cdots r_i)$ can we figure this out?

if $p(r_1 \cdots r_{i+1}) \geq p(r_1 \cdots r_i)$ $\forall i$

then $p(r_1 \cdots r_m) \geq p(r_1 \cdots r_{m-1}) \geq \ldots \geq pr(r_1) \geq pr(\Lambda) (\geq 2/3 ?)$

this is a leaf
so value is 1 or 0,
but if $\geq 2/3$
must be 1

fraction of good paths

main issue: how do we figure out the best setting of r_{i+1} at each step?

An example: Max Cut (another way to derandomize)

recall algorithm:

flip n coins $r_1 \cdots r_n$

put node i in S if $r_i = 0$ & T if $r_i = 1$

output S, T

derandomization:

expected size of cut

$$e(r_1 \cdots r_i) = E_{R_{i+1} \cdots R_N}[\,|\operatorname{cut}(S,T)| \text{ given } r_1 \cdots r_i \text{ choices made}]$$

$e(\Lambda) \geq \frac{|E|}{2}$ (from previous lecture)

no choices fixed

how do we calculate $e(r_1 \cdots r_{i+1})$?

Let

$S_{i+1} = \{ \overset{\text{nodes}}{j} \mid j \le i+1 \quad r_j = 0 \}$

$T_{i+1} = \{ \overset{\text{nodes}}{j} \mid j \le i+1 \quad r_j = 1 \}$ $\Big\}$ S + T so far

$U_{i+1} = \{ \overset{\text{nodes}}{j} \mid j \ge i+2 \ \& \ j \le n \}$ "undecided"

so

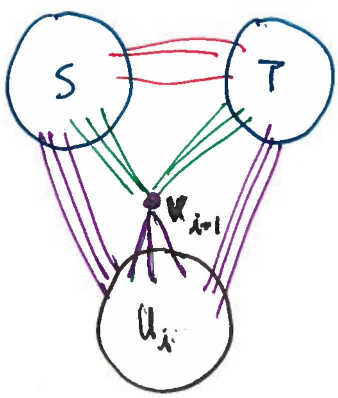
fact: $e(r_1 \cdots r_{i+1}) = (\#\text{ edges between } S_{i+1} + T_{i+1}) + \frac{1}{2}(\#\text{edges touching } U_{i+1})$

(follows from same argument as last lecture)

Note: don't need to calculate $e(r_1 \cdots r_{i+1})$

just need to compare $e(r_1 \cdots r_i\, 0)$ vs. $e(r_1 \cdots r_i\, 1)$ – is it $\{<, >\}$

note:
- # edges between S_{i+1} + T_{i+1} same for both
- U_i term is same for both



- first term differs only on edges adjacent to node $i+1$

to maximize this, place node $i+1$ to maximize cut size

i.e. $|\#\text{ edges between node } i+1 + S_i|$

vs. $|$ " " " " $+ T_i|$

Corresponds to:

<u>Greedy Algorithm</u>:

1) $S \leftarrow \phi,\ T \leftarrow \phi$

2) For $i = 0 .. N-1$

place node i in S if #edges between i & T $\geq$ # edges " i & S

else place in T

Random walks

Markov chains :

Ω = set of "states" (or nodes) (here always FINITE)

$X_0 \dots X_t \in \Omega$ sequence of visited states

Markovian property :

$$\Pr[X_{t+1}=y \mid X_0=x_0, X_1=x_1, X_2=x_2, \dots, X_t=x_t]$$
$$= \Pr[X_{t+1}=y \mid X_t=x_t]$$

Next step depends only on where you are. Not ho[w] you got there

Wlog, assume transitions independent of time :

i.e. $P(x,y) = \Pr[X_{t+1}=y \mid X_t=x]$

so can use "transition matrix" to represent it

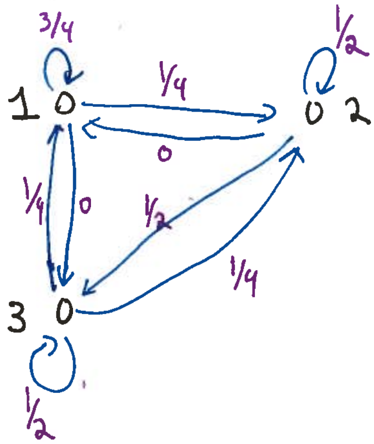
Important special case :

transitions uniform on subset corresponding to neighbors of node

def. random walk on $G=(V,E)$ is a sequence $S_0 S_1 \dots$ of nodes where S_0 is a start node. At each step i, S_{i+1} picked uniformly from $\underbrace{N(S_i)}_{\text{outedges}}$

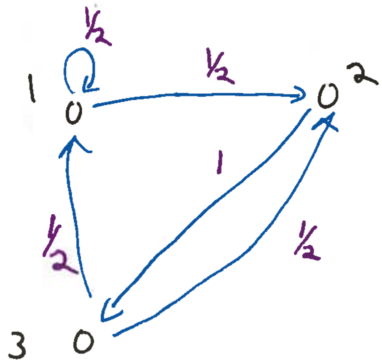
examples

Markov chain



$$P: \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 3/4 & 1/4 & 0 \\ 2 & 0 & 1/2 & 1/2 \\ 3 & 1/4 & 1/4 & 1/2 \end{array}$$

random walk on digraph



$$P: \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1/2 & 1/2 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1/2 & 1/2 & 0 \end{array}$$

$d(i) = \#$ outedges of node i

$$p(i,j) = \begin{cases} \frac{1}{d(i)} & \text{if } (i,j) \in E \\ 0 & \text{o.w.} \end{cases}$$

$$\forall i \quad \sum_j P(i,j) = 1$$

Distributions after t steps

Transition probabilities for t steps: $P^t(x,y) = \begin{cases} P(x,y) & t=1 \\ \sum_z P(x,z)P^{t-1}(z,y) & t>1 \end{cases}$ } matrix multiplication $P^t = \underbrace{P \times P \times \dots \times}_{t \text{ times}}$

Initial distribution: $\pi^{(0)} = (\pi_1^{(0)} \dots \pi_n^{(0)})$ where $\pi_i^0 = \Pr[\text{start at node } i]$

distribution after one step:

$$\pi^{(1)} = \pi^{(0)} \cdot P = \left(\sum_z P(z,1)\cdot\pi(z), \sum_z P(z,2)\pi(z), \dots\right)$$

$\vdots$

t-step distribution: $\pi^{(t)} = \pi^{(0)} \cdot P^t$

Finite Markov Chain Properties

stochastic matrix: rows of P sum to 1 — all M.C.'s have this property

doubly stochastic matrix: rows & columns sum to 1 — not even all interesting M.C.'s satisfy this

e.g. random walk on undirected ~~d-reg graphs~~ or digraph in which indegree = outdegree = const for all nodes

irreducible: ("strongly connected")

$\forall x,y \ \exists t = t(x,y)$ s.t. $P^t(x,y) > 0$

↕ change of quantifier order

ergodic: $\exists t_0$ s.t. $\forall t > t_0 \ \forall x,y \quad P^t(x,y) > 0$ ← stronger than irreducible! why?

Aperiodic:

gcd of "possible" cycle length =1

$\forall x \quad \gcd\{t : p^t(x,x) > 0\} = 1$

<u>not</u> bipartite, k-partite...

<u>Thm</u> Ergodic $\Leftrightarrow$ Irreducible + Aperiodic

Stationary Distributions

does it depend on π_0? $\Big\{$ stationary distribution π

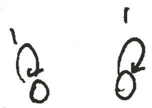
$\forall y \quad \pi(y) = \sum_x \pi(x) P(x,y)$ $\Big\}$ so $\pi^{(t)} = \pi^{(t-1)}$

Will consider P s.t. π is unique & exists $\}$ i.e. doesn't dep on π_0

if periodic: could have no stat. dist. or several

if $\pi_0 = (0,1)$ then $\pi_{2i} = (0,1)$, $\pi_{2i+1} = (1,0)$

if reducible: could have lots of stat. dist.



Some stat dists:
$(\frac{1}{2}, \frac{1}{2})$ $(0,1)$ $(1,0)$...

<u>Important Thm</u> every ergodic M.C. has unique stationary distribut