The Lovász Local Lemma

Another way to argue that "nothing bad happens"

If $A_1 \dots A_n$ are "bad" events

how do we know if there is positive probability that none occur? (or is prob that any occurs <1?)

usual way: Union bnd

no assumptions on A_i's w.r.t. independence

$Pr[\cup A_i] \leq \sum Pr[A_i]$

if each A_i occurs with prob p,

then need $p < \frac{1}{n}$ to get anything interesting (i.e. sum <1)

if A_i's independent + "nontrivial": ← "nontrivial" ≡ "$Pr(A_i) \neq 1$"

$$Pr[\cup A_i] \leq 1 - Pr[\cap \bar{A_i}]$$
$$= 1 - \prod \underbrace{Pr(\bar{A_i})}_{>0}$$
$$<1$$

ALWAYS!!

What if A_i's have "some" independence?

def A "independent" of $B_1 \dots B_k$ if $\forall J \subseteq [k]$, $J \neq \phi$

$$Pr[A \cap \bigcap_{j \in J} B_j] = Pr[A] \cdot Pr[\bigcap_{j \in J} B_j]$$

<u>def.</u> $A_1 \cdots A_n$ events
$D = (V, E)$ with $V = [n]$ is
"dependency digraph of $A_1 \ldots A_n$"
if each A_i independent of all A_j that don't neighbor it in D (ie., all A_j st. $(i,j) \notin E$)

<u>Lovász Local Lemma</u> (symmetric version)

$A_1 \cdots A_n$ events st. $pr(A_i) \le p$ $\forall i$
with dependency digraph D s.t. D is of degree $\le d$.
If $ep(d+1) \le 1$ then

$$Pr\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$$

Application:

new: degree bound restric

<u>Thm</u>. $S_1 \cdots S_m \subseteq X$, $|S_i| = \ell$, each S_i intersects at most d other S_j's

before $m < 2^{\ell-1}$ now m not restricted

if $e(d+1) \le 2^{\ell-1}$
then can 2-color X st. each S_i not monochromatic

ie. $\mathcal{H}$ is a hypergraph with m edges, each containing ℓ nodes + each intersecting $\le d$ other edges

Pf. color each elt of X red/blue with prob $\frac{1}{2}$ iid.

$A_i \equiv$ event that S_i monochromatic

$p = \Pr[A_i] = 2^{-(\ell-1)}$

A_i ind of all A_j s.t. $S_i \cap S_j = \emptyset$

depends on $\leq d$ other A_j

Since $ep(d+1) = e \frac{1}{2^{\ell-1}}(d+1) \leq 1$

LLL $\Rightarrow \exists$ 2-coloring

Comparison:

edges $= m$
size of edge $= \ell$

$m < 2^{\ell-1}$

edges $= m$
size of edge $\geq \ell$

each edge intersects $\leq d$ others

$d+1 \leq \frac{2^{\ell-1}}{e}$

no dependence on m

A second application:

Given CNF formula s.t. ℓ vars in each clause

& each var in $\leq k$ clauses.

If $\frac{e(\ell k+1)}{2^{\ell-1}} \leq 1$ there is a satisfying assignment

How do you find a solution?

partial history:

Lovász 1975 non-constructive (no fast algorithm to find soln) $d \le 2^{\ell-}$

Beck 1991 randomized algorithm but for more restrictive conditions on parameters $d \le 2^{\ell/4}$

$d \le 2^{\ell/8} \ldots d \le$

⋮

Moser 2009 negligible restrictions for SAT $d \le 2$

" " " most problems

Moser Tardos

⋮

Moser Tardos
Thm: given $S_1 \ldots S_m \subseteq X$
each S_i intersects $\le d$ other S_j's
if $e(d+1) \cdot c \le 2^{\ell-1}$
then can find 2-coloring of X s.t.
each S_i not monochromatic
in time poly in $m, d, |X|$

Moser Tardos Algorithm:
(1) 2-color all elts of X randomly
(2) While there is a monochromatic set:
- pick arbitrary monochromatic S_i
- randomly reassign colors to elements of S_i

Special Case & slower algorithm: (based on Beck + Alon)

Stronger Assumption:

let $D = d(d-1)^3$

$l = l_1 + l_2 + l_3$

$16\, D(1+d) < 2^{l_1}$ (1)

$16\, D(1+d) < 2^{l_2}$ (2)

$(2e(1+d) < 2^{l_3})$

for today, assume l is <u>constant</u>

Algorithm: Given $S_1 \cdots S_m \subseteq X$

First Pass:

For each $j \in X$

if j is "saved" do nothing

else pick color $\in \{red, blue\}$ via coin flip

Consider all S_i containing j

if S_i has l_1 pts all same color

+ <u>no</u> pts in other color

then S_i becomes dangerous

+ all uncolored pts become "frozen"

(by now, all pts in X are $\in \{red, blue, frozen\}$)

If S_i not yet 2-colored then S_i "survives"

Second Pass:

Find coloring of surviving S_i via brute force

Big Questions:

(1) Does it work?

(2) Runtime?

Analysis:

Consider a single S_i

can also happen if lots of pts are saved by nbrs.

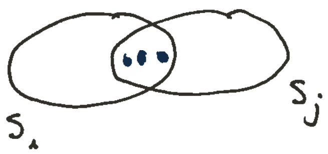
$$\Pr[S_i \text{ survives}] \geq \Pr[S_i \text{ becomes dangerous}]$$

$$= \frac{2}{2^{\ell_i}} = 2^{1-\ell_i}$$

all red or all blue

When is survival of S_i + S_j independent?

not if (1) $S_i \cap S_j \neq \emptyset$



(2) $S_i \cap S_k \neq \emptyset$ + $S_k \cap S_j \neq \emptyset$

S_k freezes pts in S_i + S_j ⇒ S_i not 2-

(3) $\exists\, k, \ell$ s.t.

$S_i \cap S_k \neq \emptyset$

$S_k \cap S_\ell \neq \emptyset$

$S_\ell \cap S_j \neq \emptyset$

Cause S_k + S_ℓ to freeze pts in S_i + S_j ⇒ S_i + S_j can't be 2-colored

Useful Graph G:

$G \leftarrow$ nodes $V \leftarrow [m]$ (node $i \sim S_i$)
edges $(i,j) \in E$ iff $S_i \cap S_j$

note: S_i only depends on other S_j within distance ≤ 3

Observe: solution exists via LLL by condition 1 after 1st pass, soln exists for survivors via condition 2

Claim: (with high prob)
all surviving nodes in G form components of size $O(\log m)$

Corr: if ℓ constant:
2^{nd} pass needs only $O(2^{\ell \cdot c \cdot \log m}) = O(m^{\ell \cdot c})$
$= \text{poly}(m)$ time

What if ℓ not constant?
recurse on components

Proof of Claim:

What is prob that component C survives?

if C survives, consider
biggest $T \subseteq C$ s.t.

(1) all nodes in T are dist ≥ 4 in G
(2) if connect nodes in T of dist $= 4$ then T connected

picking T greedily $\Rightarrow$ $\exists\, T$ of size $\geq \frac{|C|}{d^3}$

if C survives, $T \subseteq C$ also survives

$\Rightarrow$ $\Pr[T \text{ survives}] \geq \Pr[C \text{ survives}]$

What is $\Pr[T \text{ survives}]$?

$\forall\, S_i \in T$, S_i survives if
(1) dangerous
(2) <u>next</u> to dangerous $S_{i'}$ which froze its elements

<u>note</u> if $S_i \neq S_j$
then $S_{i'} \cap S_{j'} = \emptyset$ since S_i + S_j are dist 4

For each $S_i \in T$
pick S_i' possible dangerous set from S_i, nbr of S_i $\Big\}$ $(d+1)^k$ ways to make this choice

all S_i' are disjoint

$\Pr[\text{all } k\ S_i' \text{ become dangerous}] \leq 2^{(1-\ell_1)\cdot k}$

So $\Pr[\text{all } S_i \text{ survive}] \leq (d+1)^k \cdot 2^{(1-\ell_1)\cdot k}$

We need to show no such large tree survives.

here is the win!

if T was arbitrary set of size u, we have $\binom{m}{u}$ choices of T & need to show all don't survive (lots of term in union bnd)

here T is an arbitrary subtree of size u in a graph $G^{(4)}$ of degree $\le D = d(d-1)^3$.

unlabelled deg $\le D$ trees of size u is $\le 4D^u$ →

labelled trees of size u in degree D graph is $\le \underset{\uparrow}{m}\,(4D)^u$ (compare to m^u above)

initial pts for root

Expected # Trees of size u that survive

$$\le \underbrace{m(4D)^u}_{\#\text{ terms in union bnd}} \cdot \underbrace{(d+1)^u\, 2^{(1-\ell_1)\cdot u}}_{\text{prob of survival of each one}} = m\,[\underbrace{8D(d+1)2^{-\ell_1}}_{<1/2}]^u$$

$\Rightarrow$ if $u \ge \Omega(\log m)$ this term is $o(1)$

If $o(1)$ u-trees survive in expectation, then Markovs $\neq \Rightarrow$

$\Pr[\text{more than } \underbrace{k \cdot o(1)}\ u\text{-trees survive}] < \frac{1}{k}$

pick k so that this is <1

So $\Pr[\text{any } u\text{-tree survives}] < 1/k$ (note: if no u tree survives $\Rightarrow$ no $u+1$ tree survives)