Monotonicity in L_1 distance

Problem: given $f:[n]\to[0,1]$

Pass f if monotone

Fail if f is ε-far in L_1-distance from any monotone fctn.

How does it compare to Hamming distance?

when $f:[n]\to\{0,1\}$, Hamming distance equals L_1 distance

for $f:[n]\to[0,1]$, $HD \ge L_1$-dist

for $f:[n]\to[0,\dots,d]$ $HD\cdot d \ge L_1$-dist

Thm Can test if f monotone with respect to L_1-distance in $O(\frac{1}{\varepsilon})$ queries

compare to $O(\log n)$ for Hamming distance !!!

Pf.

def. for $t\in[0,1]$, threshold fctn $f_{(t)}:[n]\to\{0,1\}$,

$$f_{(t)}(x)=\begin{cases}1 & \text{if } f(x)\ge t\\ 0 & \text{o.w.}\end{cases}$$

e.g.

(if $f(0)=0$):

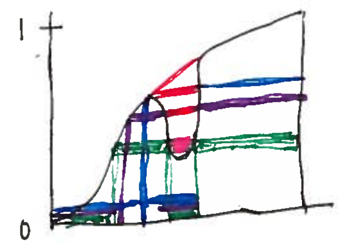
$$f(x) = \int_0^{f(x)} dt = \int_0^1 f_{(t)}(x)\, dt$$

$$= \int_0^{f(x)} \underbrace{f_{(t)}(x)}_{=1} dt + \int_{f(x)}^1 \underbrace{f_{(t)}(x)}_{=0} dt$$

(See page 2a ~~for~~ a picture)

$L_1(f, \mathcal{M}) \equiv L_1$ dist of f to closest monotone fctn

$$d_{\mathcal{M}}(f) = \frac{L_1(f, \mathcal{M})}{n}$$



Lemma $\quad d_{\mathcal{M}}(f) = \int_0^1 d_{\mathcal{M}}(f_{(t)})\, dt$

Pf. to show $L_1(f, \mathcal{M}) = \int_0^1 L_1(f_{(t)}, \mathcal{M})\, dt$

to see $L_1(f, \mathcal{M}) \le \int_0^1 L_1(f_{(t)}, \mathcal{M})\, dt$:

$\forall\, t$, g_t is closest monotone Boolean fctn to $f_{(t)}$

$g \equiv \int_0^1 g_t\, dt$ also monotone since sum of monotone fctns,

So $L_1(f, \mathcal{M}) \le \|f - g\|_1$

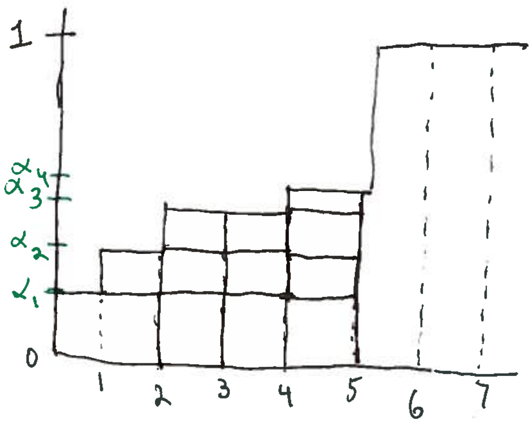
$$= \left\| \int_0^1 f_{(t)}\, dt - \int_0^1 g_t\, dt \right\|_1$$

$$= \left\| \int_0^1 f_{(t)} - g_t\, dt \right\|_1$$

$$\le \int_0^1 \|f_{(t)} - g_t\|\, dt = \int_0^1 L_1(f_{(t)}, \mathcal{M})\, dt,$$

2a
LP

$$f_t(x) = \begin{cases} 1 & \text{if} \quad f(x) \geq t \\ 0. & \text{o.w.} \end{cases}$$



$$\alpha_4 = f(4) = \underbrace{f_{(\alpha_1)}(4)}_{1}(\alpha_1 - 0) + \underbrace{f_{(\alpha_2)}(4)}_{1}(\alpha_2 - \alpha_1) + \underbrace{f_{(\alpha_3)}(4)}_{1}(\alpha_3 - \alpha_2) + \underbrace{f_{(\alpha_4)}(4)}_{1}(\alpha_4 - \alpha_3) + \sum_{i>4} \underbrace{f_{(\alpha_i)}(4)}_{0} \cdot (\alpha_i - \alpha_{i-1})$$

$$= \alpha_1 + (\alpha_2 - \alpha_1) + (\alpha_3 - \alpha_2) + (\alpha_4 - \alpha_3) + 0$$

$$= \alpha_4$$

to see $L_1(f,\mathcal{M}) \ge \int_0^1 L_1(f_{(t)},\mathcal{M})\,dt$:

g is closest mon fctn to f w.r.t. L_1

so $g_{(t)}$ mon $\forall t \in [0,1]$

$$
\begin{aligned}
L_1(f,\mathcal{M}) &= \|f-g\|_1 \\
&= \left\| \int_0^1 f_{(t)} - g_{(t)}\,dt \right\|_1 \\
&= \sum_{x: f(x) \ge g(x)} \int_0^1 f_{(t)}(x) - g_{(t)}(x)\,dt + \sum_{x: f(x) < g(x)} \int_0^1 g_{(t)}(x) - f_{(t)}(x)\,dt \\
&= \int_0^1 \sum_{x: f(x) \ge g(x)} f_{(t)}(x) - g_{(t)}(x)\,dt + \sum_{x: f(x) < g(x)} g_t(x) - f_t(x)\,dt \\
&= \int_0^1 \|f_{(t)} - g_{(t)}\|_1\,dt \ge \int_0^1 L_1(f_{(t)},\mathcal{M})\,dt
\end{aligned}
$$

since $f(x) \ge g(x)$ iff $\forall t$ $f_{(t)}(x) \ge g_{(t)}(x)$

$\square$

<u>Lemma</u> if T is nonadaptive, 1-sided error (always passes monotone f) ε-test for $f: D \to \{0,1\}$ w.r.t. Hamming
$\Rightarrow$ " is " " " " for $f: D \to [0,1]$ w.r.t. L_1

Pf.

if $f : D \to [0,1]$ mon:

T queries pts on Q + accepts iff f monotone on Q

$\Rightarrow T$ always accepts mon fctn f ✓

if $d_M(f) \geq \varepsilon$:

Lemma $\Rightarrow \exists\ t^* \in [0,1]$ s.t. $d_M(f_{(t^*)}) \geq \varepsilon$

but $f_{(t^*)}$ is Boolean s.t. $d_M(f_{(t^*)})$ = Hamming dist of $f_{(t^*)}$ from mon

$\Rightarrow T$ should fail $f_{(t^*)}$ for $\geq 3/4$ choices of Q

i.e. if Q contains $x \leq y$ s.t. $f_{(t^*)}(x) > f_{(t^*)}(y)$

but does this x, y pair also fail f?

Yes! $f(x) \geq t^* > f(y)$ so if Q fails $f_{(t^*)}$ it also fails f

$\Rightarrow T$ fails f with prob $\geq 3/4$ ∎