6.842 Randomness and Computation April 10, 2014

Homework 9

Lecturer: Ronitt Rubinfeld *Due Date: April 17, 2014*

Homework guidelines: You may work with other students, as long as (1) they have not yet solved the problem, (2) you write down the names of all other students with which you discussed the problem, and (3) you write up the solution on your own. No points will be deducted, no matter how many people you talk to, as long as you are honest. If you already knew the answer to one of the problems (call these "famous" problems), then let me know that in your solution writeup – it will not affect your score, but will help me in the future. It's ok to look up famous sums and inequalities that help you to solve the problem, but don't look up an entire solution.

The following problems are to be turned in. You should upload your solution to Stellar as a pdf file.

1. Show that the degree ≤ 1 Fourier coefficients of a halfspace uniquely determine it within the set of all Boolean functions. In other words, show that if $f : \{\pm 1\}^n \to \{\pm 1\}$ is a halfspace and $g : \{\pm 1\}^n \to \{\pm 1\}$ is a Boolean function with $\hat{f}(S) = \hat{g}(S)$ for every $S \subseteq [n]$ with $|S| \leq 1$, then $f = g$.

2. **(Almost k-wise independent random variables)** Let $\epsilon \in (0,1)$ and $k \in [n]$. A random vector $(X_1, \ldots, X_n) \in \{\pm 1\}^n$ is said to be (ϵ, k)-wise independent if the restriction of $(X_1, \ldots, X_n)$ to any subset of k coordinates in $[n]$ is ϵ-close to the uniform distribution on $\{0,1\}^k$. [1] Note that $(0,k)$-wise independence coincides with our usual notion of k-wise independence. The goal of this problem is to show that any (ϵ, k)-wise independent distribution is $O(\epsilon n^k)$- close to some k-wise independent distribution. If μ is the probability mass function (pmf) of a distribution with support $\{\pm 1\}^n$, we denote by $\hat{\mu} : \{\pm 1\}^n \to [-1, +1]$ its Fourier transform.

   (a) Show that if $(X_1, \ldots, X_n)$ is an (ϵ, k)-wise independent random vector with pmf μ, then $|\hat{\mu}(S)| \leq \epsilon / 2^{n-1}$ for every non-empty $S \subseteq [n]$ with $|S| \leq k$.

   (b) Show that if $(X_1, \ldots, X_n)$ is a random vector with pmf μ satisfying $\hat{\mu}(S) \leq \epsilon / 2^{n-1}$ for every non-empty $S \subseteq [n]$ with $|S| \leq k$, then $(X_1, \ldots, X_n)$ is $(O(2^{k/2}\epsilon), k)$-wise independent.

   (c) Show that if $(X_1, \ldots, X_n)$ is an (ϵ, k)-wise independent random vector, then it is $O(\epsilon n^k)$-close to some k-wise independent distribution.

   [Hint: Using part (b), what condition on the non-empty low-degree Fourier coefficients of μ would imply k-wise independence ? How can you modify μ in order to make it satisfy this condition for one particular Fourier coefficient ? How are the other low-degree Fourier coefficients affected ?]

[1] By saying that two distributions p and q with support $\{0,1\}^k$ are ϵ-close, we mean that the *statistical distance* $\Delta(p,q) := \frac{1}{2} \sum_{x \in \{\pm 1\}^k} |p(x) - q(x)| = \max_{S \subseteq \{0,1\}^k} (p(S) - q(S))$ is at most ϵ.