6.842 Randomness and Computation	April 3, 2014
Homework 8	
Lecturer: Ronitt Rubinfeld	*Due Date: April 10, 2014*

Homework guidelines: You may work with other students, as long as (1) they have not yet solved the problem, (2) you write down the names of all other students with which you discussed the problem, and (3) you write up the solution on your own. No points will be deducted, no matter how many people you talk to, as long as you are honest. If you already knew the answer to one of the problems (call these "famous" problems), then let me know that in your solution writeup – it will not affect your score, but will help me in the future. It's ok to look up famous sums and inequalities that help you to solve the problem, but don't look up an entire solution.

The following problems are to be turned in. You should upload your solution to Stellar as a pdf file.

1. Consider the sample complexity required to learn the class of monotone functions mapping $\{+1,-1\}^n$ to $\{+1,-1\}$ over the uniform distribution (without queries).

   (a) Show that
   $$\sum_{|S| \geq Inf(f)/\epsilon} \hat{f}(S)^2 \leq C \cdot \epsilon$$
   where $Inf(f)$ is the total influence of f (defined in Problem 2 of Homework 6), and C is an absolute constant.

   (b) Show that the class of monotone functions can be learned to accuracy ϵ with $n^{\Theta(\sqrt{n}/\epsilon)} = 2^{\tilde{O}(\sqrt{n}/\epsilon)}$ samples under the uniform distribution (where the confidence parameter δ is some small constant).
   Hint: You can use Problem 2 of Homework 6.

2. The goal of this problem is to show that for any halfspace f and any $\epsilon \in (0, 1/2]$,
   $$NS_\epsilon[f] = O(\sqrt{\epsilon})$$
   Note that this fact was used in the lecture about learning halfpsaces.

   (a) Let $A : \mathbb{N} \to \mathbb{R}$ and let C be a class of Boolean-valued functions closed under negation and under permutations of the input variables. Assume that each $f \in C$ with domain $\{\pm 1\}^n$ has $Inf(f) \leq A(n)$ where $Inf(f)$ is the total influence of f. Show that for each $f \in C$,
   $$NS_\epsilon[f] \leq \frac{1}{m} A(m)$$
   where $m = \lfloor 1/\epsilon \rfloor$ and $NS_\epsilon[f]$ denotes the noise senstivity of f with noise rate ϵ.
   Hint: One way to proceed is as follows: For every $z \in \{\pm 1\}^n$ and every hash function $\pi : [n] \to [m]$, define the function $g_{z,\pi} : \{\pm 1\}^m \to \{\pm 1\}$ by $g_{z,\pi}(w) = f(z \odot w^\pi)$ for every $w \in \{\pm 1\}^m$, where $\odot$ denotes entry-wise multiplication and $w^\pi = (w_{\pi(1)}, w_{\pi(2)}, \ldots, w_{\pi(n)}) \in \{\pm 1\}^n$. Then, show that $\frac{1}{m} E_{z,\pi}[Inf(g_{z,\pi})] = NS_{1/m}[f]$.

(b) A Boolean function is said to be *unate* if it can be turned into a monotone function by possibly negating some of its input variables. Show that every unate function with domain $\{\pm 1\}^m$ has total influence at most $\sqrt{m}$.
Hint: You can use Problem 2 of Homework 6.

(c) Use parts (a) and (b) to conclude that for every halfspace f,

$$NS_\epsilon[f] = O(\sqrt{\epsilon})$$