

Homework 4

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Fun problem, not to be turned in

A *vertex cover* V' of a set of edges E' is a set of nodes such that every edge of E' is adjacent to one of the nodes in V' .

For graph $G = (V, E)$, let the *transitive closure graph* $TC(G)$ be the graph $G^{(tc)}(V, E^{(tc)})$ where $(u, v) \in E^{(tc)}$ if there is a directed path from u to v in G .

Let $f : V \rightarrow \{0, 1\}$ be a labeling of the vertices of a known directed acyclic graph G by 0 and 1. For any pair of nodes x and y , we say that $x \leq_G y$ if there is a path from x to y in G . We say that f is *monotone* if for all $x \leq_G y$, $f(x) \leq f(y)$. The *minimum distance of f to monotone* is the minimum number of nodes that must be relabeled in order to turn f into a monotone function.

Let E' be the set of violating edges in $TC(G)$ according to f . Show that the minimum distance of f to monotone is equal to the minimum size of a vertex cover of E' .

Homework problems (to be turned in)

1. This problem is about testing monotonicity of functions defined over a directed graph G . The function maps nodes into binary values (i.e., $f : V \rightarrow \{0, 1\}$), and we say that it is *monotone* if for all directed edges (u, v) , we have that $f(u) \leq f(v)$. We say that f is ϵ -close to monotone if there is a monotone function g such that g and f differ on at most $\epsilon|V|$ entries. A testing algorithm knows the graph G in advance, and for a given node u , may query $f(u)$ in one time step.

Note, in the following, you may use the results of the fun problem above without justification.

- (a) Let $V = \{v_1, \dots, v_n\}$. For each directed graph $G = (V, E)$, let $B_G = (V', E')$ be the bipartite graph where $V' = \{v_1, \dots, v_n\} \cup \{v'_1, \dots, v'_n\}$, and $(v_i, v'_j) \in E'$ iff v_j is reachable from v_i in G .
Show that a q -query testing algorithm for f over graph B_G with distance parameter $\epsilon/2$ yields a q -query testing algorithm for f over graph G with distance parameter ϵ .
 - (b) Let f be a function on V which is ϵ -far from monotone over graph G . Then $TC(G)$ has a matching of violated edges of size at least $(\epsilon/2)|V|$. (Recall previous problem).
 - (c) Show that if f is a function over bipartite graph G (where all edges are directed from “left-to-right”, that is from v_i to v'_j), there is a test for monotonicity of f with query complexity at most $O(\sqrt{|V|/\epsilon})$.
2. Let $L = \{uu^r vv^r \mid u, v \in \{0, 1\}^*, 2(|u| + |v|) = n\}$. We saw in class that given a string x , distinguishing $x \in L$ from x that is ϵ -far (meaning that $> \epsilon n$ bits of x need to be changed in order to make x a member of L) requires $\Omega(\sqrt{n})$ queries. Give an algorithm for this problem that uses $O(\sqrt{n} \log n / \text{poly}(\epsilon))$ queries to the input. The running time does not have to be sublinear.