6.842 Randomness and Computation September 11, 2017

Lecture 2: Lovász Local Lemma and Moser-Tardos Algorithm

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1 Lovász Local Lemma

Definition 1 (independence)**.** An event A is *independent* of $\{B_1, \dots, B_k\}$ if for any nonempty $J \subseteq [k]$,

$$\Pr[A \cap \bigcap_{j \in J} B_j] = \Pr[A] \Pr[\bigcap_{j \in J} B_j].$$

Definition 2 (dependency digraph)**.** Let $V = [n]$. The dependency digraph $D = (V, E)$ of events $A_1, \dots, A_n$ is defined such that if node i has outgoing edges to $j_1, \dots, j_d$, A_i is independent of $\{A_j \mid j \in [n] \backslash \{j_1, \dots, j_d\}\}$.

Theorem 3 (symmetric Lovász Local Lemma)**.** *Let $A_1, \dots, A_n$ be events such that $\Pr[A_i] \leq p$ for all i, and their dependency graph has max degree d. If $\mathrm{e}p(d+1) \leq 1$, then $\Pr[\bigcap_{i=1}^n \bar{A}_i] > 0$.*

In other words, we can view A_i as the bad events, and there is a nonzero probability that none of them occur if $p \leq \frac{1}{\mathrm{e}(d+1)}$. Unlike the union bound, this depends on d but not n. Below we describe its applications and the Moser-Tardos algorithm, which can be considered as a constructive proof.

2 Application to Two-Coloring

Theorem 4. *Let $S_1, \dots, S_m \subseteq S$ with $|S_i| = l$, and S_i intersects at most d other S_j's. If $\mathrm{e}(d+1) \leq 2^{l-1}$, S can be two-colored such that each S_i is not monochromatic.*

Proof. Color each element of S as black or white each with probability 1/2. Let A_i be the event that S_i is monochromatic, which occurs if and only if its l elements are all black or all white, i.e.,

$$p = \Pr[A_i] = 2^{-l} + 2^{-l} = 2^{1-l}.$$

Since A_i is dependent on A_j if and only if S_i intersects S_j, and S_i intersects at most d other S_j's, the dependency graph of $A_1, \dots, A_m$ has max degree d. Since

$$\mathrm{e}p(d+1) = e2^{1-l}(d+1) \leq 1,$$

by Lovász Local Lemma $\Pr[\bigcap_{i=1}^n \bar{A}_i] > 0$, i.e., S can be two-colored such that each S_i is not monochromatic.

3 Moser-Tardos Algorithm

Theorem 5. *Let $S_1, \dots, S_m \subseteq S$ with $|S_i| = l$, and S_i intersects at most d other S_j's. If $c\mathrm{e}(d+1) \leq 2^{l-1}$ for some constant $c > 1$, S can be two-colored such that each S_i is not monochromatic in expected $O(\mathrm{poly}(m, l))$ time.*

3.1 Algorithm

The algorithm is simple and intuitive: assign each element either color with probability 1/2, and while there is some monochromatic S_i, reassign either color to each element of S_i with probability 1/2.

The algorithm is correct as long as it terminates. Below we bound its expected runtime.

3.2 Runtime Analysis

To facilitate the analysis, we define the *log of execution* and the *witness trees*, and finish the analysis in next lecture.

Definition 6. The *log of execution* is an sequence of pairs $(1, S_{r_1}), (2, S_{r_2}), \ldots$, where S_{r_j} is the recolored set in the j-th loop of the algorithm.

Definition 7. The *witness tree* T_j for loop j begins with root T_{r_j}, and for $t = j-1, j-2, \ldots, 0$, if S_{r_t} intersects some node in the current T_j, add S_{r_t} to be a child of some node that intersects S_{r_t} such that it has maximum depth. (If there are multiple choices, pick one arbitrarily).