6.842 Randomness and Computation	September 18, 2017
Homework 2	
Lecturer: Ronitt Rubinfeld	*Due Date: September 25, 2017*

Homework guidelines: You may work with other students, as long as (1) they have not yet solved the problem, (2) you write down the names of all other students with which you discussed the problem, and (3) you write up the solution on your own. No points will be deducted, no matter how many people you talk to, as long as you are honest. If you already knew the answer to one of the problems (call these "famous" problems), then let me know that in your solution writeup – it will not affect your score, but will help me in the future. It's ok to look up famous sums and inequalities that help you to solve the problem, but don't look up an entire solution.

The following problems are for your understanding only, do not turn them in:

1. **(Moser-Tardos algorithm)** Show that in the Moser-Tardos algorithm, there is a constant c such that the probability that there exists a set that gets resampled more than $c \log m$ times is bounded by 1/10.

The following problems are to be turned in. Please turn in each solution on a separate piece of paper.

1. Let k, m satisfy $e(m(m-1)+1)k(1-1/k)^m \leq 1$. Let $S \subset \mathcal{Z}$ with $|S| = m$ and $T \subset \mathcal{Z}$ with $|T|$ finite. Show that there exists a k-coloring $\chi : \mathcal{Z} \to [k]$ such that every translate $S+t$ is k-colored. That is, for all $t \in T$ and $1 \leq i \leq k$, there exists $s \in S$ with $\chi(s+t) = i$.
2. **(Directed cycles)** Let $D = (V, E)$ be a simple directed graph (that is, a directed graph with no self-loops and with at most one edge between ever pair of vertices). Assume that D has minimum outdegree δ and maximum indegree Δ. Show that if $e(\Delta\delta+1)(1-\frac{1}{k})^\delta < 1$, then D contains a (directed, simple) cycle whose length is a multiple of k.
   Hint: Let $f : V \to \{0, 1, \ldots, k-1\}$ be a random coloring of V obtained by choosing for each $v \in V$, $f(v) \in \{0, \ldots, k-1\}$ independently according to a uniform distribution. For each $v \in V$, consider the event A_v that there is no $u \in V$ s.t. $(v, u) \in E$ and $f(u) = f(v) + 1 \bmod k$.