Decoding Error-Correcting Codes via Linear Programming

Ph.D. Thesis Defense

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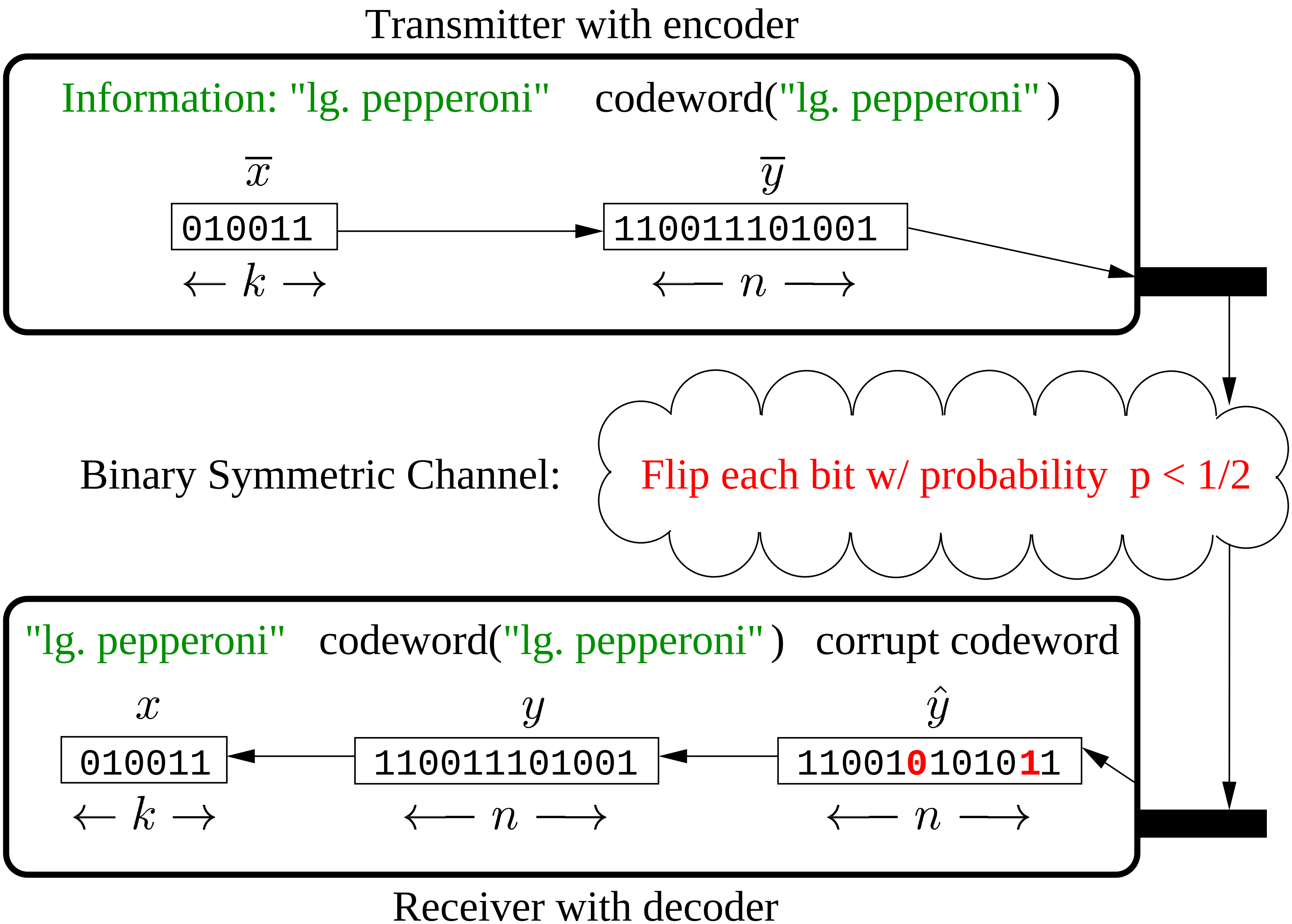
Binary Error-Correcting Code



Repetition Code Example

- Encoder: Repeat each information bit 5 times.

  Information word: `1011`

  Codeword: `11111 00000 11111 11111`

  Corrupt codeword: `10110 01000 01001 10111`

- Decoder: Take majority witihin every group of 5.

  Decoded codeword: `11111 00000 00000 11111`

  Decoded info word: `1001`

- Information transmitted successfully

  $\iff$ at most 2 bits flipped per group of 5.

Outline

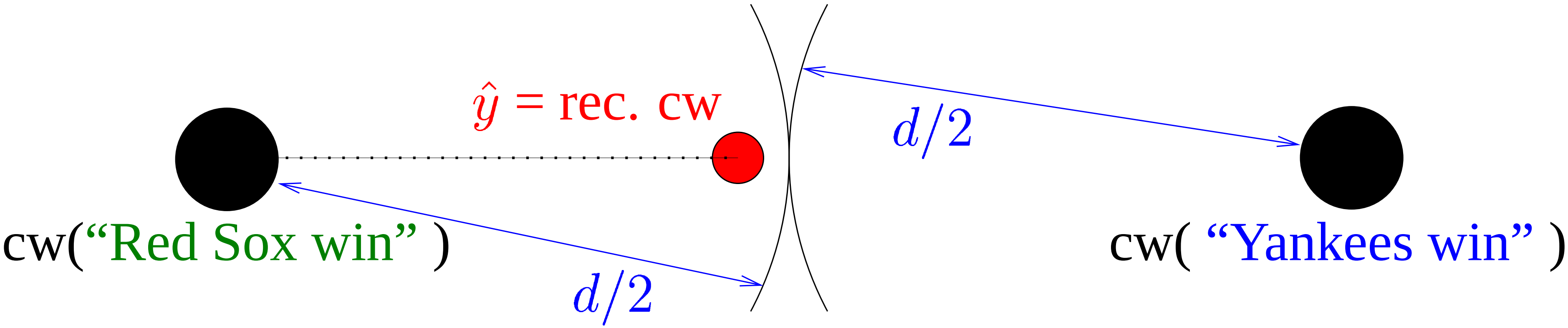
- Coding background
- Our contributions:
  - LP decoding: general method.
  - LP decoders for turbo, LDPC codes.
  - Performance bounds tor turbo, LDPC codes.
  - Connections to message-passing decoders.
  - Experiments (supporting theory).
  - Methods for tightening the relaxation.
  - New dual-based message-passing algorithms.
- Future work

Basic Coding Terminology

- A code is a subset $C \subseteq \{0,1\}^n$, where $|C| = 2^k$.
- Block length = length = n. Affects latency, encoder/decoder complexity, performance.
- Rate = k/n. Measures redundancy of transmission. Affects efficiency, performance.
- Minimum distance = distance = $d = \min_{y,y' \in C} \Delta(y, y')$. "Worst case" measure of performance.
- Word error rate (WER) = probability of decoding failure = $\Pr_{\text{noise}}[\text{transmitted } \overline{y} \neq \text{ decoded } y]$. Practical measure of performance.

Maximum-Likelihood (ML) Decoding

- ML decoders minimize WER.
  - BSC: Finds $y \in C$ s.t. $\Delta(y, \hat{y})$ is minimum.
  - Corrects errors up to half the minimum distance.

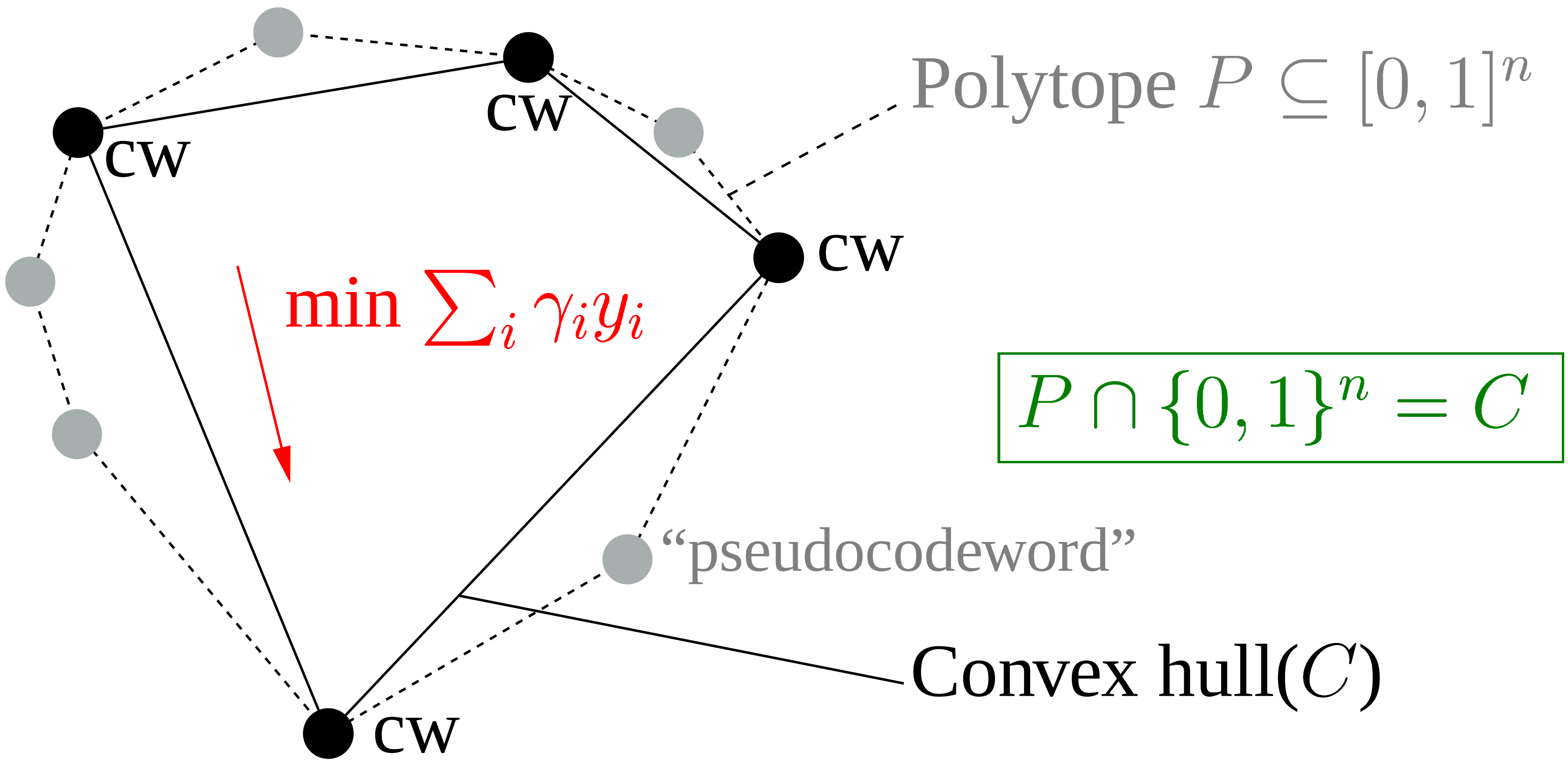


- Cost function γ_i: *negative log-likelihood ratio* of y_i.

$$[\gamma_i > 0 \implies y_i \text{ likely } 0] \quad [\gamma_i < 0 \implies y_i \text{ likely } 1]$$

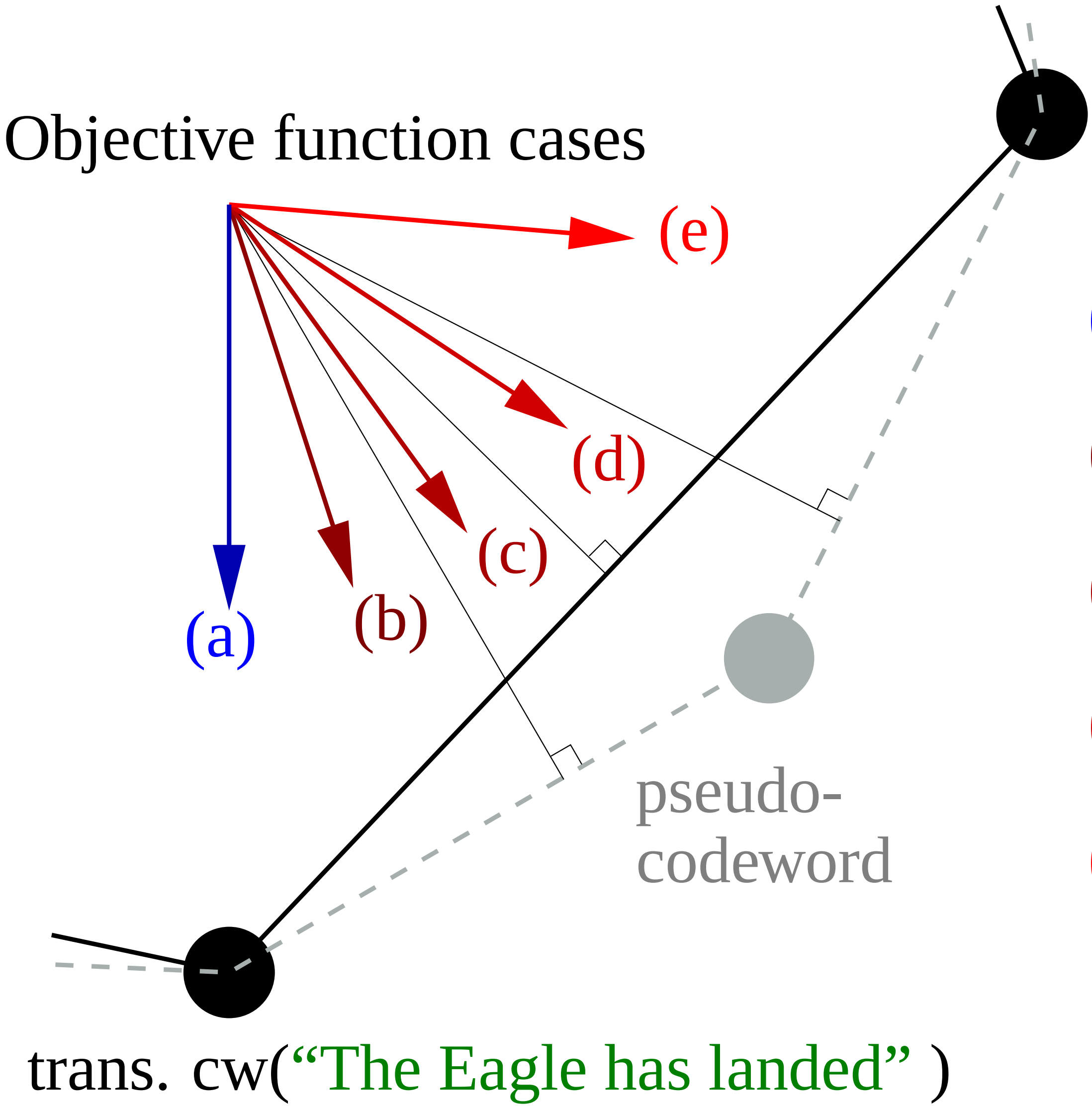
> ML DECODING: Given corrupt codeword $\hat{y}$,
> find $y \in C$ such that $\sum_i \gamma_i y_i$ is minimized.

LP Decoding



- LP variables y_i for each code bit, relaxed $0 \leq y_i \leq 1$.
- Alg: Solve LP. If y^* integral, output y^*, else "error."
- *ML certificate* property

LP Decoding Success Conditions



(a) No noise

(b) Both succeed

(c) ML succeed, LP fail

(d) Both fail, detected

(e) Both fail, undetected

Fractional Distance

- Another way to define (classical) distance d:
  - d = min l_1 dist. between two integral vertices of P.
- Fractional distance:
  - d_{frac} = min l_1 distance between an integral vertex and any other vertex of P.
  - Lower bound on classical distance: $d_{frac} \leq d$.

Theorem: In the binary symmetric channel, LP decoders can correct up to $\lceil d_{frac}/2 \rceil - 1$ errors.

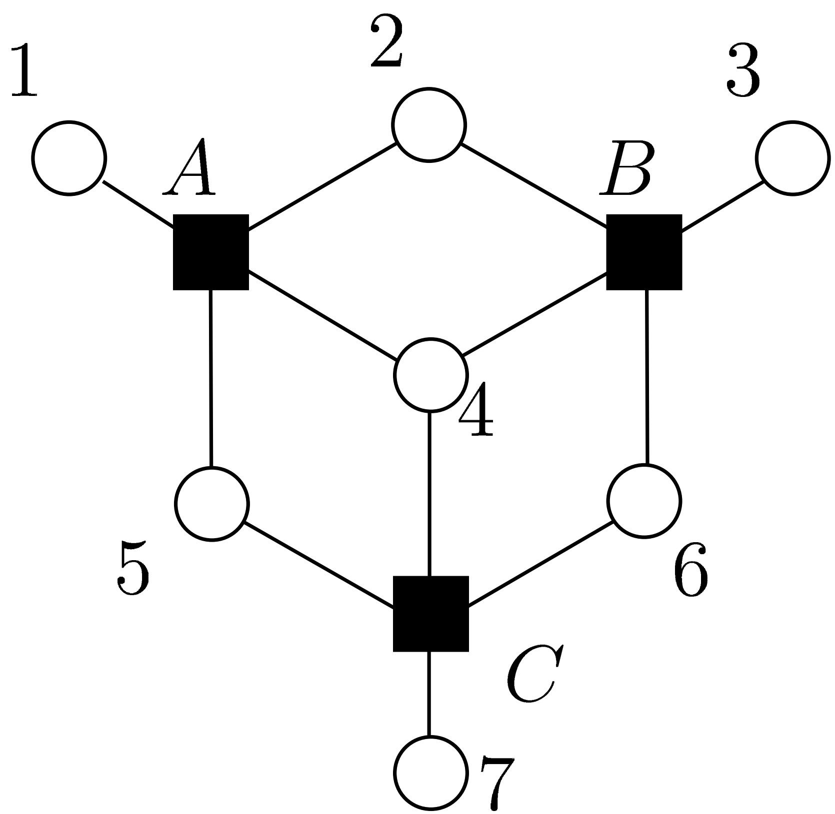
- Given facets of P, fractional distance can be computed efficiently.

Turbo Codes + LDPC Codes

- Low-Density Parity-Check (LDPC) codes [Gal ’62] .
- Turbo Codes introduced [BGT ’93], unprecedented error-correcting performance.
- Ensuing LDPC “Renaissance:”
  - Expander codes [SS ’94]
  - Message-passing algorithms [Wib ’96]
  - Connection to belief-propagation [MMC ’98]
  - Message-passing capacity [RU, LMSS, RSU, BRU, CFDRU, ’99-’01]
  - Designing irregular codes [LMSS ’01]
  - Connection to “Bethe free energy” [Yed ’02]

Factor Graph

- *Factor (Tanner) Graph* of a linear code: bipartite graph modeling the *parity check matrix* of the code.

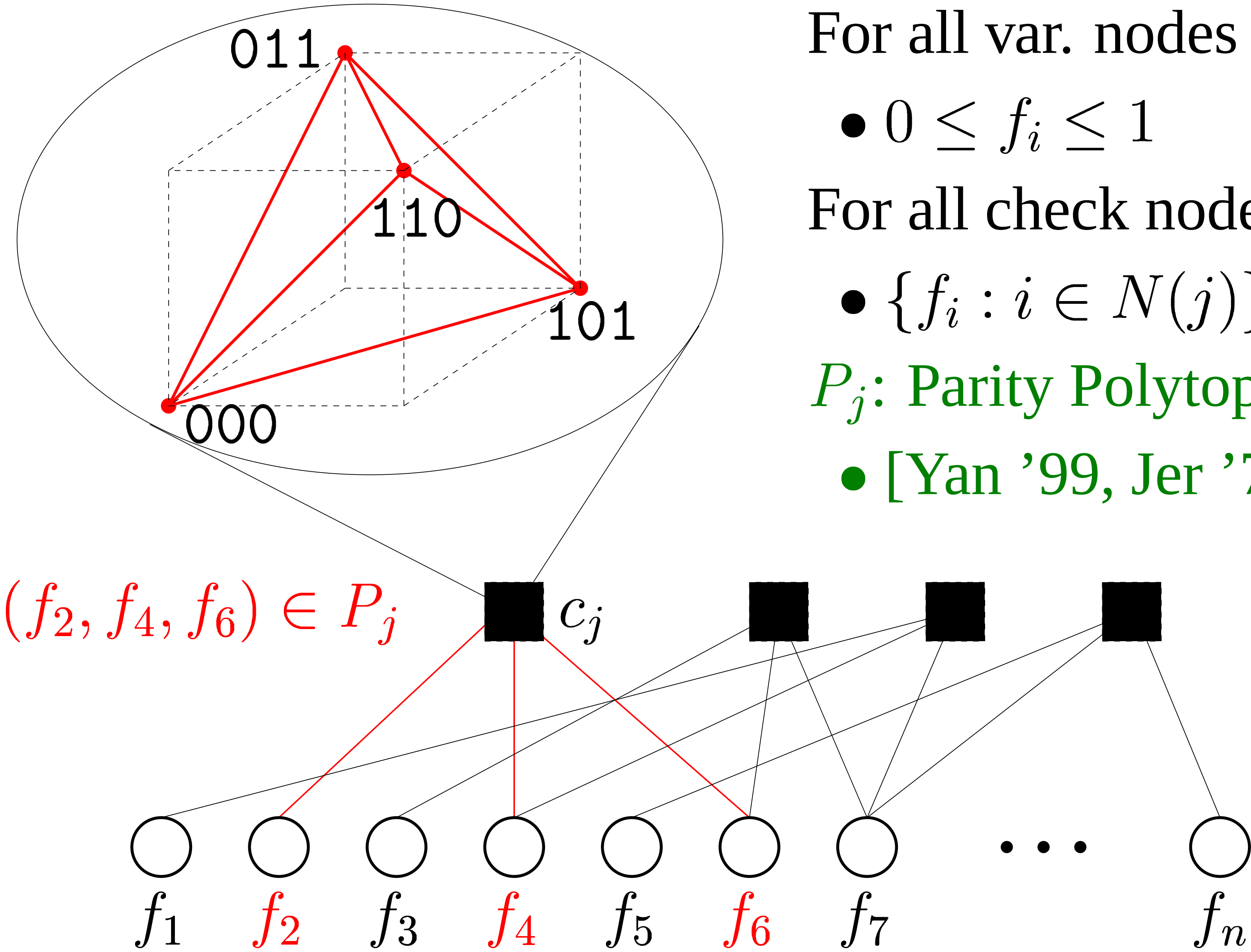


- "Variable nodes" $y_1, \ldots, y_n$.
- "Check Nodes" $c_1, \ldots, c_m$.
- $N(j)$: neighborhood of check c_j.
- Codewords: $y \in \{0,1\}^n$ s.t.:

$$\forall c_j, \sum_{i \in N(j)} y_i = 0 \pmod 2$$

- Codewords: 0000000, 1110000, 1011001, etc.

LP Relaxation on the Factor Graph



For all var. nodes i:

- $0 \leq f_i \leq 1$

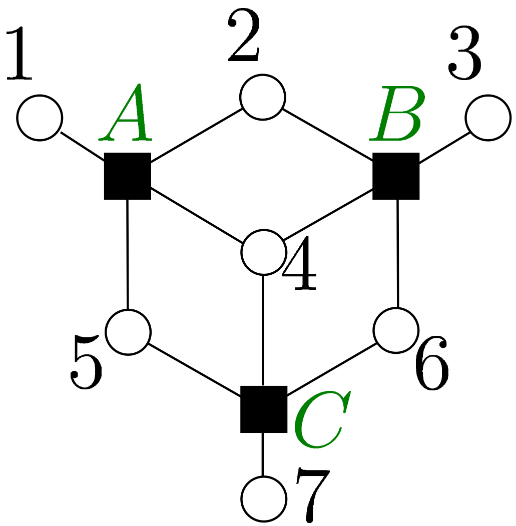
For all check nodes j:

- $\{f_i : i \in N(j)\} \in P_j$

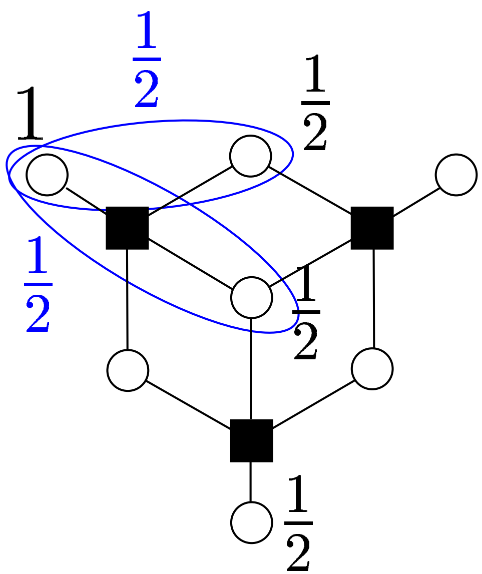
P_j: Parity Polytope

- [Yan '99, Jer '75]

Fractional Solutions



- Suppose:
$$\begin{aligned}\gamma_1 &= -2.8\\ \gamma_2 &= +0.8\\ \gamma_{3..7} &= +1\end{aligned}$$
- ML codeword: $[1, 1, 1, 0, 0, 0, 0]$
- ML codeword cost: -1.



- Frac. sol: $f = [1, \frac{1}{2}, 0, \frac{1}{2}, 0, 0, \frac{1}{2}]$.
- Satisfies LP constraints?

A: $[1, \frac{1}{2}, \frac{1}{2}, 0] = \frac{1}{2}[1, 1, 0, 0] + \frac{1}{2}[1, 0, 1, 0]$

B,C: similar.

- Frac. sol cost: -1.4.

LP Decoding Success Conditions

- Pr[Decoding Success] = Pr[$\overline{y}$ is the unique OPT].
- Can we assume $\overline{y} = 0^n$? (This is a common asssumption for linear codes.)

Thm: For LP decoding of binary linear codes, the WER is independent of the transmitted codeword.

- Pr[$\overline{y}$ is the unique OPT] = Pr[All pcw's cost > 0].
- "Combinatorial" characterization of pseudocodewords (scale the LP vertices).

Thm: The LP decoder succeeds iff all non-zero pseudocodewords have positive cost.

Performance Bounds

- **Turbo Codes:** For rate-1/2 RA (cycle) codes: If G has large girth, negative-cost points in P are rare.
  - Erdös (or [BMMS '02]): Hamiltonian 3-regular graph with girth $\log n$.

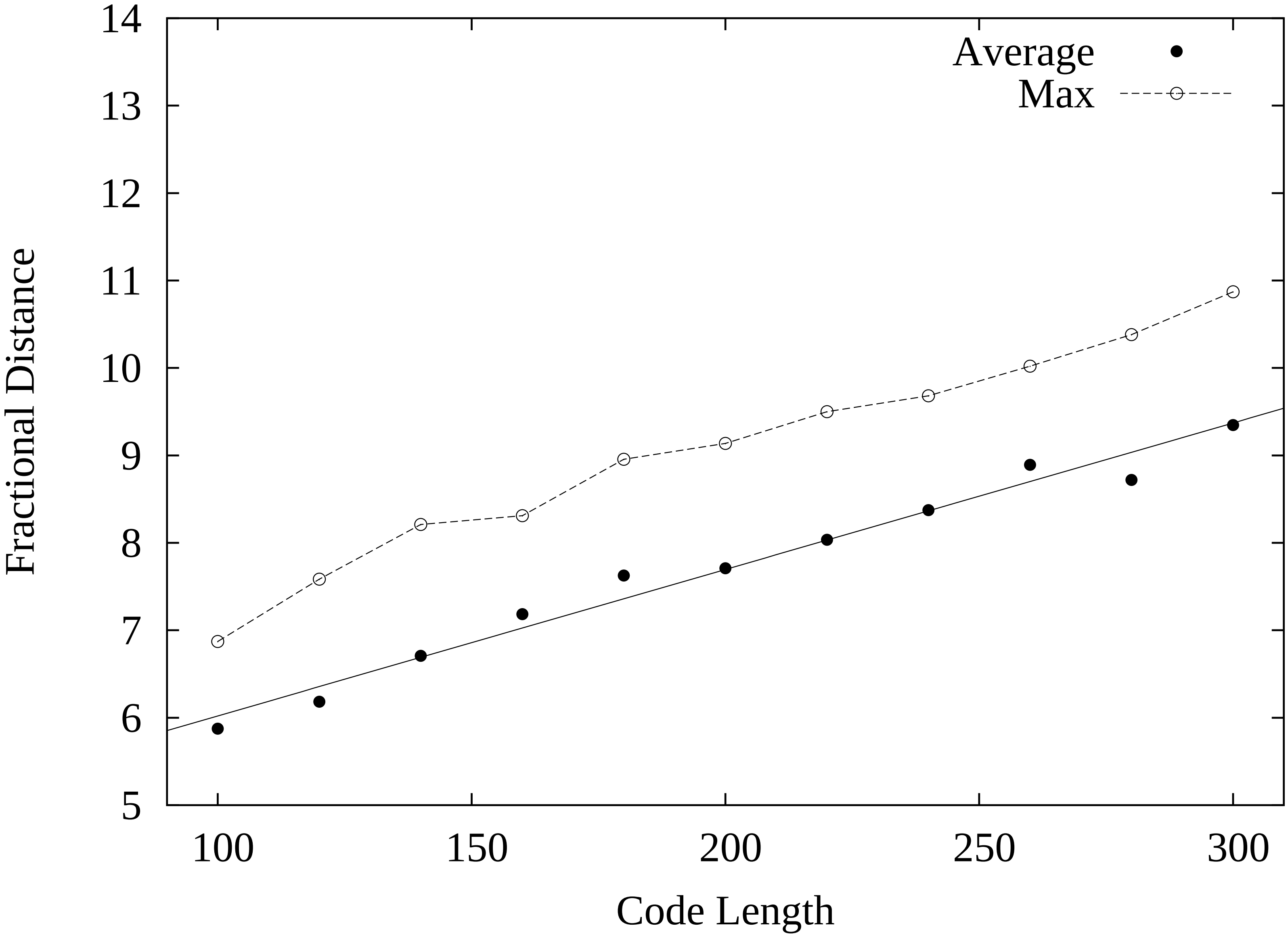
Thm: For any $\alpha > 0$, if $p < 2^{f(\alpha)}$, then WER $\leq n^{-\alpha}$.

- **LDPC Codes:** All var. nodes in G have degree $\geq d_\ell$:

Thm: If G has girth g, then $d_{frac} \geq (d_\ell - 1)^{\lceil g/4 \rceil - 1}$

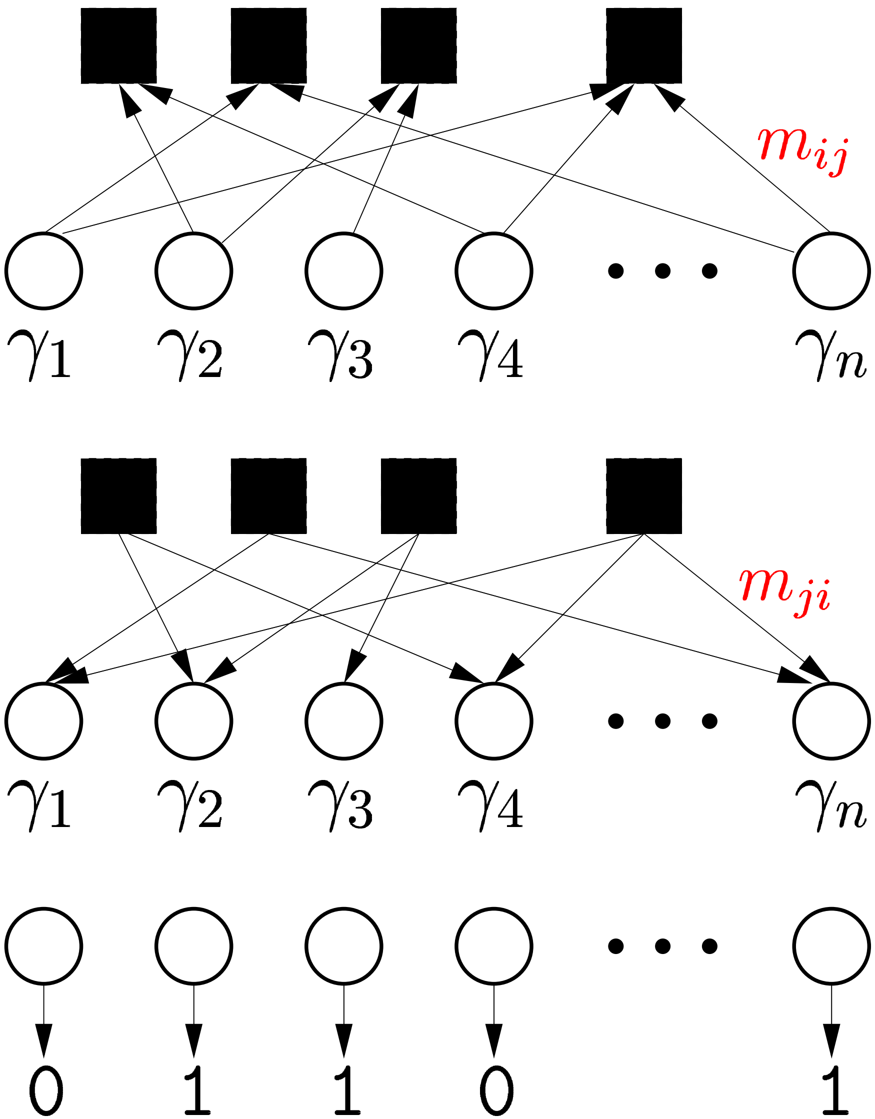
  - Can achieve $d_{frac} = \Omega(n^{1-\epsilon})$. Stronger graph properties (expansion?) are needed for stronger results.

Growth of Fractional Distance



- Random (3,4) LDPC Code

Message-Passing Decoders



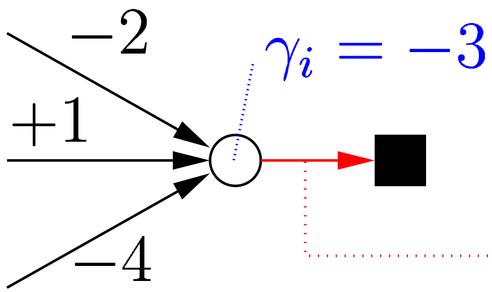
(a) Var-to-check messages

(b) Check-to-var messages

Repeat

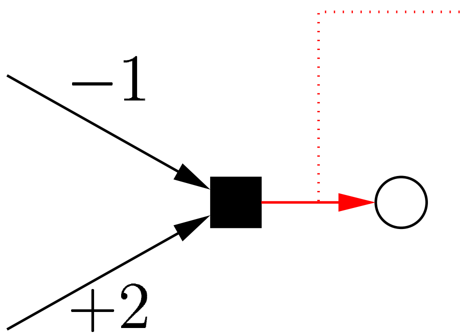
(c) Hard Decision

Min-Sum Update Rules



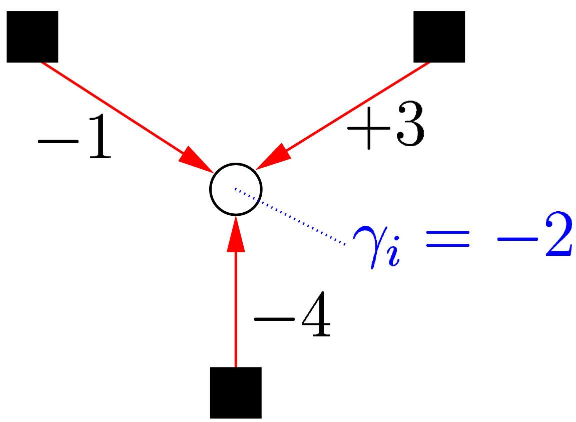
$$m_{ij} = \gamma_i + \sum m_{ji} = -8$$

$$m_{ji} = \min(S : i = 1) - \min(S : i = 0)$$



$[1,0,1] : -1$ $[0,1,1] : +2$	$[0,0,0] : 0$ $[1,1,0] : +1$

$$-1 \quad - \quad 0 \quad = -1$$



- Let $x = \sum m_{ji} + \gamma_i$.
  - if $x > 0$, output 0
  - if $x < 0$, output 1

Analyzing Message-Passing Decoders

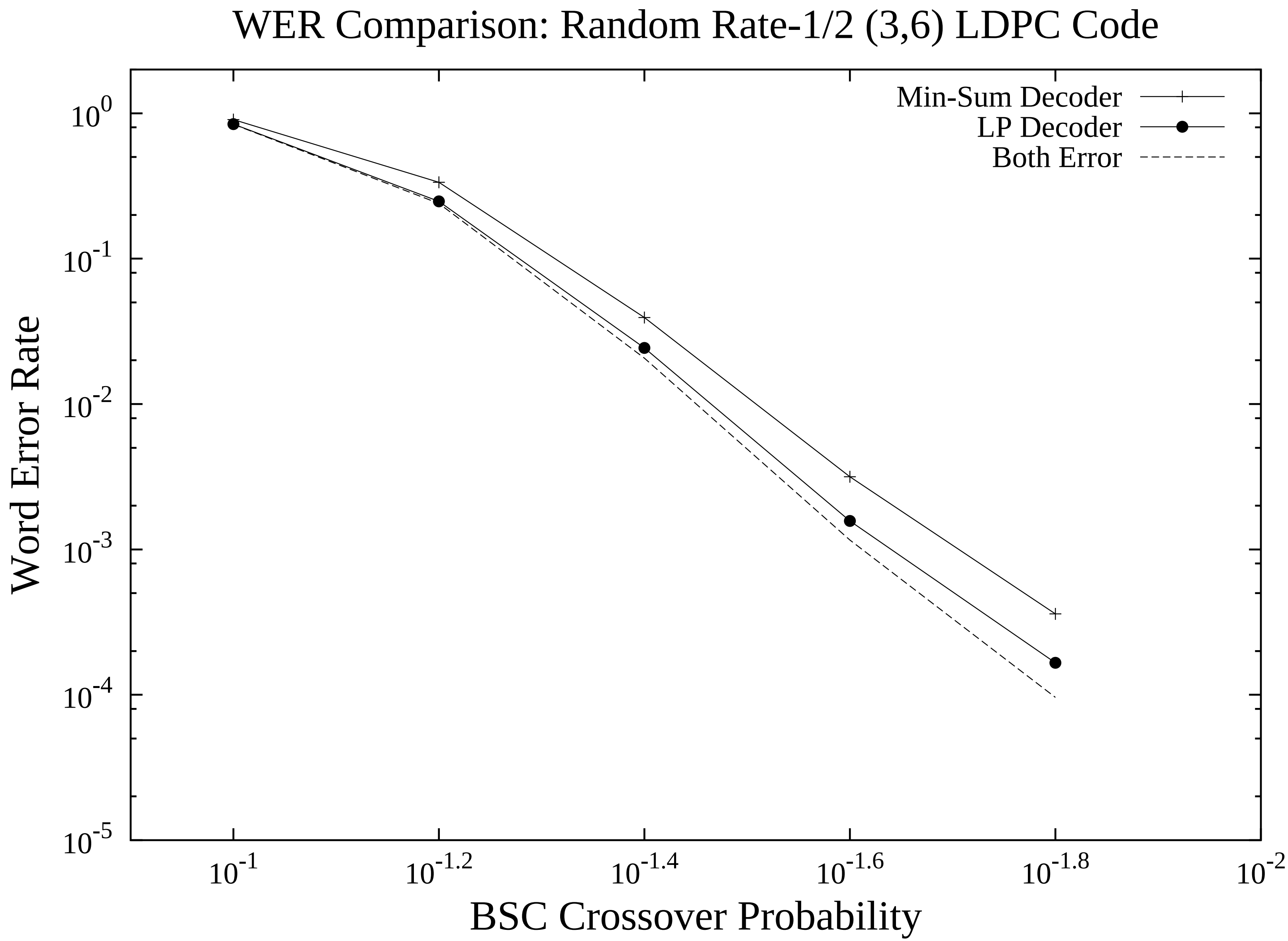
- Sum-product, min-sum, Gallager, Sipser/Spielman, tree-reweighted max-product [WJW '02].
- Message cycles: dependencies difficult to analyze.
- Density Evolution [RU '01, LMSS '01, ...]:
  - Assume "tree-like" message neighborhood, random graph from ensemble.
  - If err < threshold, any WER achievable (with high probability), for sufficiently large n.
- Finite-length analysis: combinatorial error conditions known for the binary erasure channel [DPRTU '02].
- LP Decoding: well-characterized error conditions for general channels, any block length, even with cycles.

Unifying other "pseudocodewords"

- BEC: Sum-prod. fails $\iff$ *stopping set* [DPTRU '02].
  - **Thm: LP pseudocodewords = stopping sets.**
- Tail-Biting trellisses: Min-sum fails $\iff$ neg-cost *dominant pseudocodeword* [FKMT '98].
  - **Thm: LP pcws. = dominant pseudocodewords**
- Cycle Codes: Min-sum fails $\iff$ neg-cost *irreducible closed walk* [Wib '96].
  - **Thm: LP pcws. = irreducible closed walks**

- LDPC codes: Min-sum fails $\iff$ neg-cost *deviation set* in computation tree [Wib '96].
  - **LP pseudocodewords**: natural "closed" analog of deviation sets.

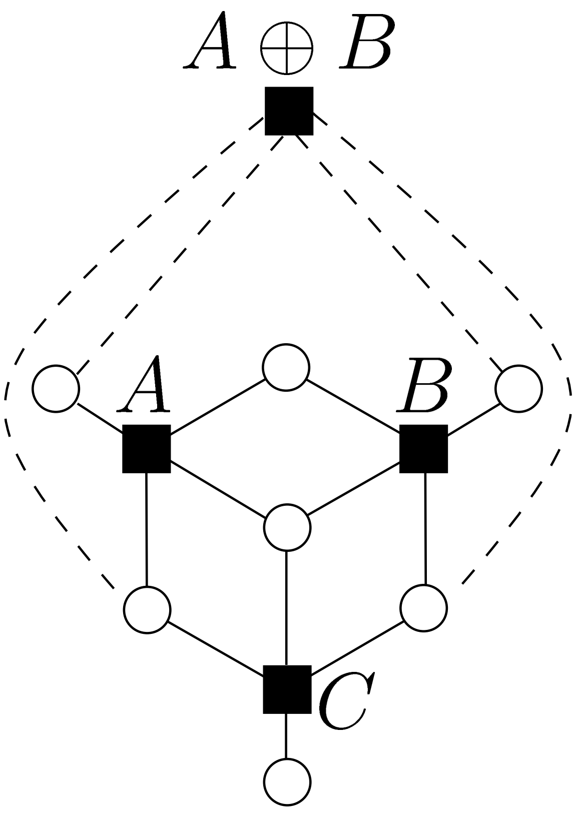
Performance Comparison

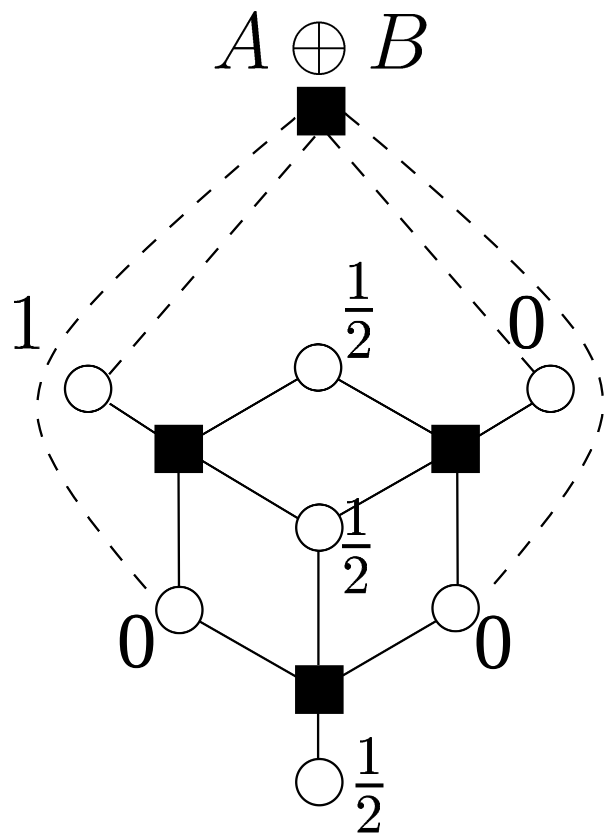


- Length 200, left degree 3, right degree 6.

Tightening the Relaxation

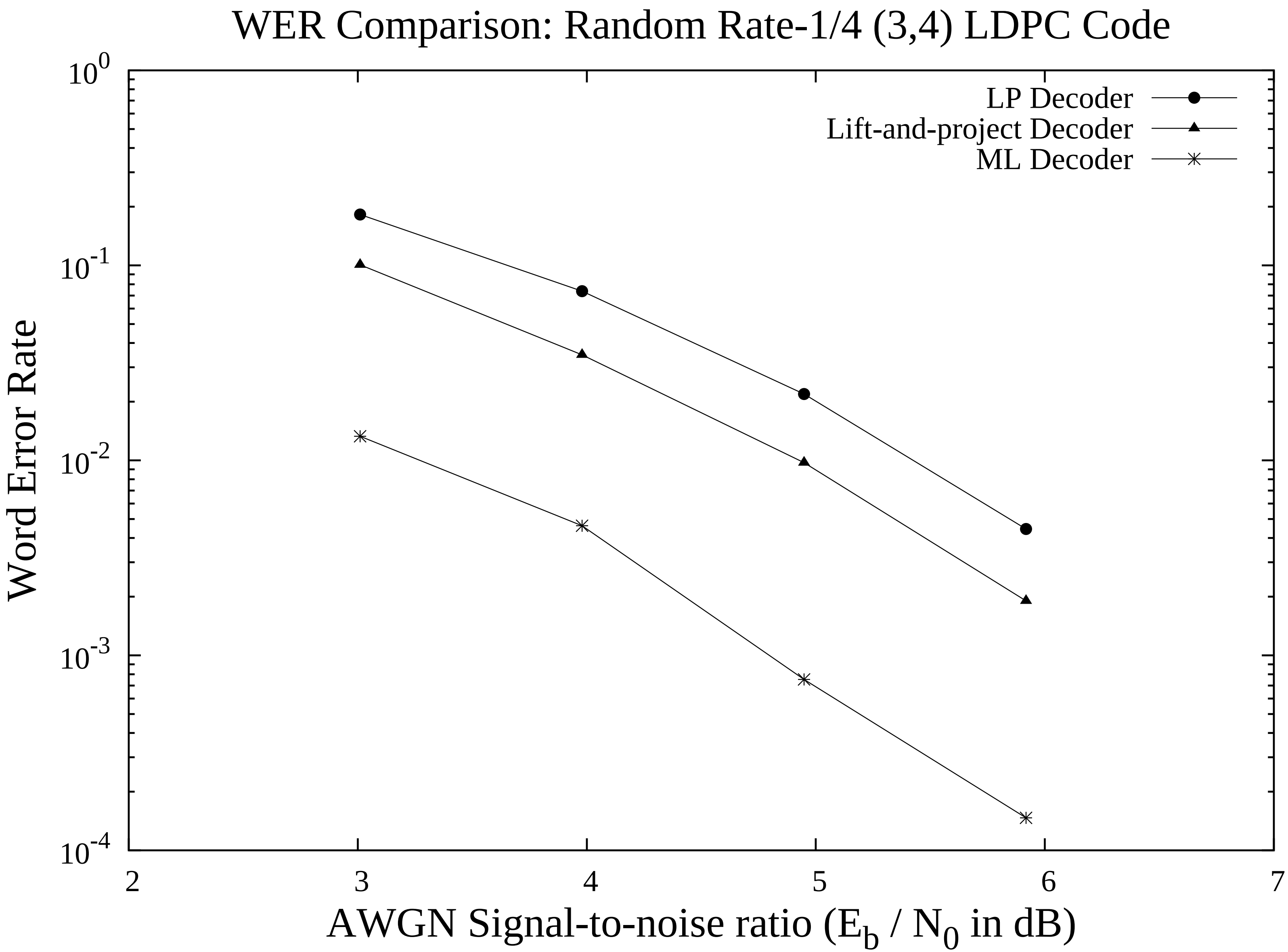
- If constraints are added to the polytope, the decoder can only improve. Example: redundant parity checks.





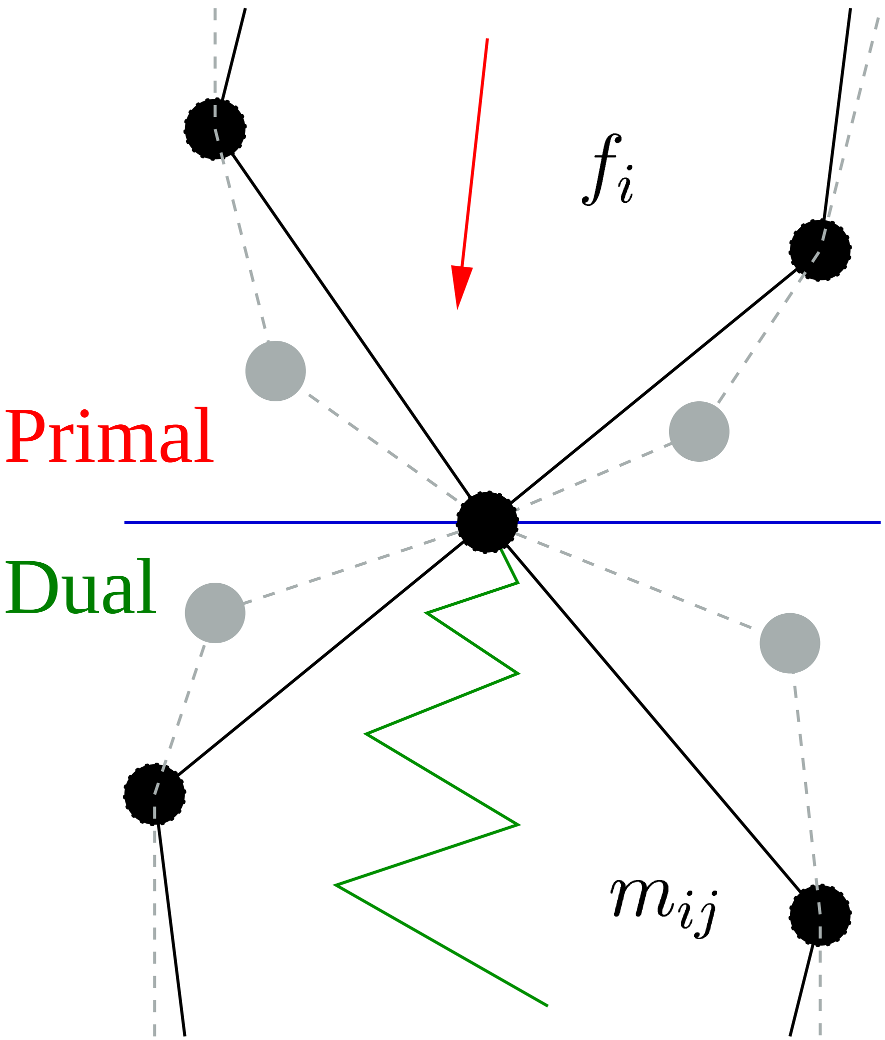
- Generic tightening techniques [LS ’91] [SA ’90].

Using Lift-And-Project



- Length 36, left degree 3, right degree 4.

New Message-Passing Algorithms



$\leftarrow$ Original LP relaxation

- Dual variables: messages.
- Enforce dual constraints.
- Convergence to codeword $\implies$ primal optimum.
- ML certificate.

New Message-Passing Algorithms

- Tree-reweighted max-product uses LP dual variables

  $\Longrightarrow$ TRMP has ML certificate property.

- Using complimentary slackness, conventional message-passing algorithms gain ability to show an ML certificate.

- Use subgradient algorithm to solve dual directly.
  - Gives message passing algorithm with ML certificate property, combinatorial success characterizations.

Future Work

- New WER, fractional distance bounds:
  - Lower rate turbo codes (rate-1/3 RA).
  - Other LDPC codes, including
    - Expander codes,
    - Irregular LDPC codes,
    - Other constructible families.
  - Random linear/LDPC codes?
- ML Decoding using IP, branch-and-bound?
- Using "lifting" procedures to tighten relaxation?
- Deeper connections to "sum-product" (belief-prop)?
- LP decoding of other code families, channel models?