Geometric Games on Triangulations
Extended Abstract

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1 Introduction

Let S be a set of n points in the plane, which we assume to be in general position, i.e., no three points of S lie on the same line. A *triangulation* of S is a simplicial decomposition of its convex hull having S as vertex set.

In this work we consider several perfect-information combinatorial games involving the vertices, edges (straight-line segments) and faces (triangles) of some triangulation. We describe main broad categories of these games and provide in various situations polynomial-time algorithms to determine who wins a given game under optimal play, and ideally, to find a winning strategy.

We present games where two players $\mathcal{R}$(ed) and $\mathcal{B}$(lue) play in turns, as well as *solitaire* games for one player. In some *bichromatic* versions, player $\mathcal{R}$ will use red and player $\mathcal{B}$ will use blue, respectively, to color some element of the triangulation. In *monochromatic* variations, all players (maybe the single one) use the same color, green.

Games on triangulations come in three main flavors:

- *Constructing (a triangulation).* The players construct a triangulation $T(S)$ on a given point set S. Starting from no edges, players $\mathcal{R}$ and $\mathcal{B}$ play in turn by drawing one or more edges in each round. In some variations, the game stops as soon as some structure is achieved. In other cases, the game stops when the triangulation is complete, the last move or possibly some counting decides then who is the winner.

- *Transforming (a triangulation).* A triangulation $T(S)$ on top of S is initially given, all edges originally being black. In each turn, a player applies some local transformation to the current triangulation, resulting in a new triangulation. The game stops when a specific configuration is achieved or no more moves are possible.

- *Marking (a triangulation).* A triangulation $T(S)$ on top of S is initially given, all edges and nodes originally being black. In each turn, some of its elements are marked (e.g. colored) in a game-specific way. The game stops when some configuration of marked elements is achieved (possibly the whole triangulation) or no more moves are possible.

For each of the variety of games described in Section 2, we are interested in characterizing who wins the game, and designing efficient algorithms to determine the winner and compute a winning strategy. More details about the games can be found in the full papers [1] and [2].

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2 Examples of Games

We describe next the rules of several specific games that we have studied. Our intention here is to make clear which kind of games we are dealing with.

2.1 Constructing

2.1.1 Monochromatic Complete Triangulation. The players construct a triangulation $T(S)$ on a given point set S. Starting from no edges, players $\mathcal{R}$ and $\mathcal{B}$ play in turn by drawing one edge in each round. Each time a player completes one or more empty triangle(s), it is (they are) given to this player and it is again her turn (an "extra move"). Once the triangulation is complete, the game stops and the player who owns more triangles is the winner.
2.1.2 Monochromatic Triangle. Starts as in 2.1.1, but has a different stopping condition: the first player who completes one empty triangle is the winner.
2.1.3 Bichromatic Complete Triangulation. As in 2.1.1, but the two players use red and blue edges. Only monochromatic triangles count.
2.1.4 Bichromatic Triangle. As in 2.1.2, but with red and blue edges. The first empty triangle must be monochromatic.

2.2 Transforming

2.2.1 Monochromatic Flipping. Two players start with a triangulation whose edges are initially black. Each move consists of choosing a black edge, flipping it, and coloring the new edge green. The winner is determined by normal play, meaning that the goal is to make the last complete move.
2.2.2 Monochromatic Flipping to Triangle. Same rules as for 2.2.1, except now the winner is who completes the first empty green triangle.
2.2.3 Bichromatic Flipping. Two players play in turn, selecting a flippable black edge e of $T(S)$ and flipping it. Then e as well as any still-black boundary edges of the enclosing quadrilateral become red if it was player $\mathcal{R}$'s turn, and blue if it was player $\mathcal{B}$'s move. The game stops if no more flips are possible. The player who owns more edges of her color wins.
2.2.4 All-Green Solitaire. In each move, the player flips a flippable black edge e of $T(S)$; then e becomes green, as do the four boundary edges of the enclosing quadrilateral. The goal of the game is to color all edges green.
2.2.5 Green-Wins Solitaire. As in 2.2.4, but the goal of the game is to obtain more green edges than black edges.

2.3 Marking

2.3.1 Triangulation Coloring Game. Two players move in turn by coloring a black edge of $T(S)$ green. The first player who completes an empty green triangle wins.
2.3.2 Bichromatic Coloring Game. Two players $\mathcal{R}$ and $\mathcal{B}$ move in turn by coloring red respectively blue a black edge of $T(S)$. The first player who completes an empty monochromatic triangle wins.
2.3.3 Four-Cycle Game. Same as 2.3.1 but the goal is to get an empty quadrilateral.
2.3.4 Nimstring Game. *Nimstring* is a game defined in Winning Ways [4] as a special case of the classic children's (but nonetheless deep) combinatorial game Dots and Boxes [3, 4]. In the context of triangulations, players in Nimstring alternate *marking* one-by-one the edges of a given triangulation (i.e., coloring green an edge, initially black), and whenever a triangle has all three of its edges marked, the completing player is awarded an extra move and must move again. The winner is determined by normal play. Thus, the player marking the last edge of the triangulation actually loses, because that last edge completes one or two triangles, and the player is forced to move again, which is impossible.

Besides beauty and entertainment, games keep attracting the interest of mathematicians and computer scientists because they also have applications to modeling several areas and because they often reveal deep mathematical properties of the underlying structures, in our case the combinatorics of planar triangulations.

Games on triangulations belong to the more general area of *combinatorial games* which typically involve two players, $\mathcal{R}$(ed) and $\mathcal{B}$(lue). A *game position* consists of a set of options for Red's moves and a set of options for Blue's moves, where each option is itself a game, representing the game position resulting from the move. We define next a few more terms from combinatorial game theory that we will use in this paper. For more information, refer to the books [4, 5] and the survey [6]. The paper [7] contains a list of more than 900 references.

We consider games with *perfect information* (no hidden information as in many card games) and there are no chance moves (like rolling dice). Most of the games we consider (the monochromatic games) are also *impartial* in the sense that the options for Red are the same as the options for Blue. In this case, a game is simply a set of games, and can thus be viewed as a tree. The leaves of this tree correspond to the empty-set game, meaning that no options can be played; this game is called the *zero game*, denoted 0.

In general, each leaf game might be assigned a label of whether the current player reaching that node is a winner or loser, or the players tied. However, a common and natural assumption is that the zero game is a losing position, because the next player to move has no move to make. We usually make this assumption, called *normal play*, so that the goal is to make the last move. In contrast, *misère play* is just the opposite: the last player able to move loses. In more complicated games, the winner is determined by comparing scores.

Any impartial perfect-information combinatorial game without ties has one of two outcomes under optimal play (when the players do their best to win): a *first-player win* or a *second-player win*. In other words, whoever moves first can force herself to reach a winning leaf, or else whoever moves second can force herself to reach a winning leaf, no matter how the other player moves throughout the game. Such forcing procedures are called *winning strategies*. For example, under normal play, the game 0 is a second-player win, and the game having a single move to 0 is a first-player win, in both cases no matter how the players move. More generally, impartial games may have a third outcome: that one player can force a tie.

The Sprague-Grundy theory of impartial games (see e.g. [4], Chapter 3) says that, under normal play, every impartial perfect-information combinatorial game is equivalent to the classic game of Nim. In (single-pile) Nim, there is a pile of $i \geq 0$ beans, denoted $*i$, and players alternate removing any positive number of beans from the pile. Only the empty pile $*0$ results in a second-player win (because the first player has no move); for any other pile, the first player can force a win by removing all the beans. If a game is equivalent to $*i$, then i is called the *Nim value* of the game.

3 Overview of Results

In this section we briefly summarize some of our results from the papers [1] and [2], where all proofs and details can be found.

Theorem 1. *Deciding whether the Triangulation Coloring Game on a simple-branching triangulation (no two inner triangles share a common diagonal) on n points in convex position is a first-player win or a second-player win, as well as finding moves leading to an optimal strategy, can be solved in time linear in the size of the triangulation.*

Theorem 2. *The Monochromatic Triangle Game on n points in convex position is an incarnation of a known game called* Dawson's Kayles *[4]. It is thus a second-player win when $n \equiv 5, 9, 21, 25, 29 \pmod{34}$ and for the special cases $n = 15$ and $n = 35$; otherwise it is a first-player win. Each move in a winning strategy can be computed in time linear in the size of the triangulation.*

Theorem 3. *The outcome of the Monochromatic Complete Triangulation Game on n points in convex position is a first-player win for n odd, and a tie for n even.*

Theorem 4. *Whether a player can win the All-Green Solitaire Game for a given triangulation of n points in convex position can be decided in time $O(n)$. When the player can win, a winning sequence of moves can be found within the same time bound.*

Theorem 5. *The player of the Green-Wins Solitaire Game can obtain from any given triangulation on n points at least $1/6$ of the edges to be green at the end of the game. There are triangulated point sets such that no sequence of flips of black edges provides more than $5/9$ of the edges to be green at the end. (In the above fractions we don't pay attention to additive constants).*

Theorem 6. *The player of the Green-Wins Solitaire Game can always win for any given triangulation on $n \geq 4$ points in convex position.*

Theorem 7. *Nimstring in a fan with an even number of vertices is a first-player win.*

Theorem 8. *Nimstring in a wheel with an odd number of vertices is a second-player win.*

Theorem 9. *Four-Cycle in a triangulation whose dual is a path is a first-player win.*

Theorem 10. *Four-Cycle in a wheel with more than four triangles is a second-player win.*

Theorem 11. *Monochromatic Flipping on top of n points in convex position is a first-player win if n is even and a second-player win if n is odd.*

Theorem 12. *There is a constant N such that Monochromatic Flipping to Triangle in a triangulation (whose dual is a path) of $n \geq N$ points in convex position is a first-player win for n even, and a second-player win for n odd.*

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