Interlocked Open Linkages with Few Joints*

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ABSTRACT

We advance the study of collections of open linkages in 3-space that may be *interlocked* in the sense that the linkages cannot be separated without one bar crossing through another. We consider chains of bars connected with rigid joints, revolute joints, or universal joints and explore the smallest number of chains and bars needed to achieve interlock. Whereas previous work used topological invariants that applied to single or to closed chains, this work relies on geometric invariants and concentrates on open chains.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms**]: Nonnumerical Algorithms—*Geometrical problems and computations*

General Terms

Theory

Keywords

Linkages, Knots, Geometry, Configurations, Robotic Arms, Protein Models

1. INTRODUCTION

Consider a simple polygonal chain that is embedded in 3-space with disjoint, straight-line edges, which we think of as fixed-length *bars*. We call a chain with k bars a k*-chain*. The $k+1$ vertices of a k-chain are the two end points, adjacent to the end bars, and $k-1$ internal vertices, or *joints*. We can place restrictions that each joint be *rigid*, permitting no relative motion between its two incident bars, or be *revolute*, a term that we will consistently use for a rotational joint that preserves the angle between its two incident bars, or be *flexible*, serving as a universal joint that allows any rotation.

A *motion* of a chain is a motion of the vertices that preserves the length of the bars, respects the restrictions on joints, and never causes nonadjacent bars to touch. We say that a collection of disjoint, simple chains can be *separated* if, for any distance d, there is a motion whose result is that every pair of points on different chains has distance at least d. If a collection cannot be separated, we say that its chains are *interlocked*.

In this paper, we characterize collections of open chains with small numbers of bars that can interlock. Our results on pairs of chains, summarized in Table 1, explore when it is possible for an open k-chain to interlock with an open m-chain. A result that an open k-chain can interlock with an m-chain also implies that open or closed l-chain, with $l > k$, can interlock with an m-chain, and a result that no open k-chain can interlock with an m-chain also implies that no open l-chain with $l < k$ can interlock with an m-chain.

In addition, we show that

- Two flexible 3-chains with any finite number of flexible 2-chains cannot interlock, but three flexible 3-chains can interlock.
- A flexible 4-chain with any finite number of flexible 2-chains cannot interlock, but a flexible 3-chain and 4-chain can interlock.

We prove results on separability of chains in Section 2, and on interlocked chains in Section 3. Our proofs assume general position, namely that no nonincident bars are coplanar and no three joints collinear. Since we can enforce general position by a small perturbation, this assumption can be made without loss of generality. We list some remaining open problems in Section 4.

Previous work has considered motions of single chains and of closed chains. A *straightening* of a flexible chain is a motion that makes all joint angles become 180°. If a single chain cannot be straightened, we say that it is *locked*. It is known that a single, open chain in 3-space, having as few as 5 bars, can be locked [4, 1]. In a companion paper [7], we showed examples with open and closed chains that were interlocked, including an open 3-chain with a quadrilateral and an open 4-chain with a triangle. In these previous works it was possible to (conceptually) close an open chain by adding a piece of rope, then argue that geometric properties kept the rope from interfering with any motion, and that topological invariants demonstrated that the resulting closed links were interlocked. However, this approach does not ex-

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		2-chain		3-chain			4-chain			5-chain
		flexible	rigid	flexible	revolute	rigid	flexible	revolute	rigid	rigid
2-chain	flexible	−	−	−	−	−[5]	−[2]	−[5]	−[5]	+[15]
	rigid	−	−	−	−[4]	+[12]	+[14]	+[14]	+[14]	+
3-chain	flexible	−	−	−[1]	−[6]	+[18]	+[11]	+	+	+
	revolute	−	−[4]	−[6]	+[21]	+	+	+	+	+
	rigid	−[5]	+[12]	+[18]	+	+	+	+	+	+
4-chain	flexible	−[2]	+[14]	+[11]	+	+	+	+	+	+
	revolute	−[5]	+[14]	+	+	+	+	+	+	+
	rigid	−[5]	+[14]	+	+	+	+	+	+	+
5-chain	rigid	+[15]	+	+	+	+	+	+	+	+

Table 1: Our results on interlocking pairs of open chains. (+) = **can,** (−) = **cannot interlock. In superscript is the number of the theorem proving the result, the other entries are implied.**

tend: we cannot simply close two or more open chains with ropes because the ropes may interfere with one another. Instead we establish geometric invariants, typically about the convex hull of joints and the relations of the end bars, often by considering convenient projections of the linkage. We emphasize the different proof techniques used within each section.

One of the inspirations for our work was a question posed by Anna Lubiw [6]: into how many pieces must a chain be cut so that the pieces can be separated and straightened? This question is motivated by proteins, which may, according to some theories, temporarily split apart in order to reach the minimum-energy folding. Our results on open flexible chains, along with the locked 5-chain of [4, 1], imply that a set of chains can always be separated and every chain straightened if the total number of middle bars is less than three. If the end bars are long enough, there are interlocked configurations whenever the number of middle bars is at least three. Soss [8] investigated revolute chains[1], also motivated by proteins, and created a "staple and hook" example of an interlocked revolute 3-chain and 4-chain. We have an interlocked example with two revolute 3-chains.

The complexity of deciding whether a given chain can be unlocked is not known. One decision procedure applies the roadmap algorithm for general motion planning [2, 3], which runs in polynomial space but exponential time. Because all of our results are for a few chains, each of a few joints, the roadmap algorithm could in principle establish interlock for our examples, but couldn't discover them and probably wouldn't give insight into their structure. On the other hand, the separability proofs apply to general classes of sets of chains, rather than the specific instances handled by the algorithm.

2. SEPARABLE CHAINS

In this section, we prove that certain configurations are separable by extending a scaling idea (whose earliest reference we know is de Bruijn [5]) and other arguments to find a separating motion. Except for a couple of cases involving a flexible 2-chain, the theorems in this section are tight in the sense that, for the chains considered, any additional bars or further restrictions on the motion can allow an interlocked configuration.

2.1 Two 3-flexible chain+many 2-flexible cannot interlock

We show that two 3-chains (even with added 2-chains) never form an interlocked configuration.

Theorem 1. *Two open, flexible 3-chains and any finite number of flexible 2-chains can always be separated.*

Proof. Consider two 3-chains C_1 and C_2, and especially their middle bars, k_1 and k_2. By our general position assumption, non-adjacent bars are not coplanar. Let K be a plane between, and parallel to, the middle links k_1 and k_2. We may choose the coordinate system such that K is the yz plane. If necessary, apply another small perturbation to ensure that no two vertices have the same x coordinates except for the vertices of k_1 and of k_2.

Now, consider the affine transformation $x \to \alpha x$ for any real $\alpha \geq 1$. Note that this is a non-uniform scaling that increases all distances between pairs of points with different x-coordinates. Thus, it preserves the lengths of k_1 and k_2, and increases the length of all the other edges.

Create a motion parameterized by time $t \geq 1$ by placing the chains according to the transform for $\alpha = t$, and truncating the edges at both ends of each chain to preserve the lengths. Because affine transformations preserve incidence relationships among lines, the motion cannot cause any bars to touch. As t becomes large, the chains separate arbitrarily far, so they are not interlocked. □

We can prove a similar theorem for an open 4-chain and 2-chains.

Theorem 2. *An open, flexible 4-chain and any finite number of flexible 2-chains can always be separated.*

Proof. As in the proof of Theorem 1, rotate the configuration so that the three joints of the 4-chain are parallel to

[1]Here, and throughout this paper, a revolute joint is one that preserves the angle between the adjacent bars, which is called an "edge spin" in [9] and a "dihedral motion" in [8]. In some areas "revolute" is used for the larger class of pin joints whose axes need not align with one of the bars; we use only the restricted definition.

the yz plane, and apply the affine transformation $x \to \alpha x$ for any real $\alpha \geq 1$ to increase the distance between all vertices except the joints of the 4-chain. Each end bar can be truncated to obtain a separating motion. □

A corollary improves the bound for a problem posed by Lubiw, and first addressed in [7].

COROLLARY 3. *Given a n-chain, it is always possible to cut* $\lfloor (n-3)/2 \rfloor$ *vertices so that the pieces obtained can be separated and straightened.*

PROOF. Cut the 4th joint, then cut every other joint to obtain one 4-chain and many 2-chains. □

The next three subsections establish theorems on pairs of chains with restricted motions.

2.2 2-rigid+3-revolute cannot interlock

THEOREM 4. *A rigid* 2*-chain and a revolute* 3*-chain cannot interlock.*

PROOF. Consider the rigid 2-chain $P = (p_0, p_1, p_2)$ and the revolute 3-chain $R = (r_0, r_1, r_2, r_3)$. The general position assumption ensures that no two non-adjacent edges are coplanar. Let H be the plane containing P. Then R intersects H in at most three points: let r'_i be the intersection between $r_i r_{i+1}$ and H, if it exists.

The two lines containing $p_0 p_1$ and $p_1 p_2$ divide H into 4 quadrants $Q_1, \ldots, Q_4$. If quadrant Q_1, Q_2, or Q_3 contains no intersection point r'_i, then P can be separated by a translation in H: if Q_1 is empty, we translate P in the direction $\overrightarrow{p_1 p_2}$, if Q_2 is empty, we translate P in the direction $\overrightarrow{p_2 p_1}$, and if Q_3 is empty, we translate P in the direction $\overrightarrow{p_0 p_1}$. Otherwise, we may assume $r'_{i_1} \in Q_1$, $r'_{i_2} \in Q_2$ and $r'_{i_3} \in Q_3$.

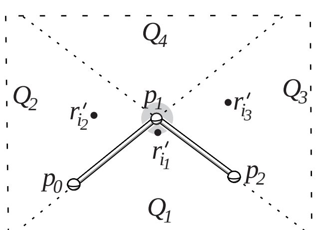

Figure 1: 2-chain P in its plane H.

Now, translate P in H so that joint p_1 is within a distance ε of r'_{i_1} as shown in Fig. 1, where $\varepsilon > 0$ is a small value to be chosen later. This can be done without intersections. If the segment $r'_{i_2} r'_{i_3}$ does not intersect P, then we can rotate P counterclockwise about r'_{i_1} until quadrant Q_2 (which is changing shape as P rotates) becomes empty—then translate P in the direction $p_2 p_1$. There remains the case in which segment $r'_{i_2} r'_{i_3}$ intersects P. We analyze two subcases: either $i_1 = 1$ and r'_1 is in Q_1, or $i_1 \neq 1$.

If $r'_1 \in Q_1$, suppose that the middle bar of R is fixed. Then the end bar $r_0 r_1$ can move in a cone with apex r_1 and axis $r_1 r_2$ passing through r'_1. If ε was chosen small enough, this cone intersects H in a curve (a conic section) that connects point r'_0 to some point in quadrant Q_4 without intersecting Q_1. Bar $r_0 r_1$ can rotate until it reaches the ray from r'_1 through $r'_{i_3} \in Q_3$ without intersecting bar $r_2 r_3$, so we can rotate r'_0 into Q_4, then can separate P by a translation in H.

For the last case, we assume without loss of generality that $r'_1 \in Q_2$, $r'_0 \in Q_1$ and $r'_2 \in Q_3$. Then, for any $\delta > 0$, we can choose ε small enough so that P can be translated to be at distance at most δ from r_1 without crossings. Because the vertex angles at r_1 and r_2 are fixed, we can choose δ small enough in order to rotate $r_2 r_3$ without crossings to bring it arbitrarily close to $r_0 r_1$. Then, for some small values of δ and then ε, the cone describing the motions of $p_1 p_2$ when $p_0 p_1$ is fixed does not intersect $r_1 r_2$, and we can move $p_1 p_2$ until we fall into one of the previous cases. □

2.3 2-flexible+4-rigid cannot interlock

When the 2-chain is flexible, the extra degree of freedom allows it to escape in its plane from any chain that intersects the plane in at most four points.

THEOREM 5. *A flexible* 2*-chain and a rigid* 3*-chain,* 4*-chain, or closed* 5*-chain cannot interlock.*

PROOF. As in the previous theorem, let the 2-chain $P = (p_0, p_1, p_2)$ define a plane H and four quadrants, $Q_1, \ldots, Q_4$. Consider the at most four points where the other chain R intersects H. If one of the quadrants Q_j, for $j \in \{1, 2, 3\}$, does not contain at least one intersection point, then we can separate P from R by translation in H.

We could move point p_1 along $\overrightarrow{p_2 p_1}$, allowing $p_0 p_1$ to rotate if it reaches any point in Q_2, unless and until p_1 approaches a ray $\rho_{12} = \overrightarrow{r_1 r_2}$, with $r_1 \in R \cap Q_1$ and $r_2 \in R \cap Q_2$. We could then move p_1 along ray ρ_{12}, until p_1 approaches a ray $\rho_{13} = \overrightarrow{r'_1 r_3}$, with $r'_1 \in R \cap Q_1$ and $r_3 \in R \cap Q_3$. If these motions do not separate the chains, then we have found two rays that cross in Q_4. This implies that $r_1 \neq r'_1$ and we know the quadrants of all four points of $R \cap H$. We can now straighten the 2-chain P by a motion in H that preserves the ray/chain intersection points $r_{12} \cap P$ and $r_{13} \cap P$. Then we can separate P from R by translation. □

2.4 3-flexible+3-revolute cannot interlock

THEOREM 6. *A flexible* 3*-chain and a revolute* 3*-chain cannot interlock.*

PROOF. Let $P = (p_0, \ldots, p_3)$ denote the flexible 3-chain and $R = (r_0, \ldots, r_3)$ denote the revolute 3-chain. Consider the projection of the two chains from the viewpoint p_1 onto a sphere. All three bars of R and $p_2 p_3$ project to segments of great arcs of angle $< \pi$, and $p_0 p_1$ and $p_1 p_2$ project to points. Thus $p_0 p_1$ can be moved arbitrarily close to $r_1 r_2$ unless its projection is enclosed in a triangle formed by $r_0 r_1$, $r_2 r_3$ and $p_2 p_3$. But then, looking at the projection from viewpoint p_2 instead, $p_2 p_3$ can be moved arbitrarily close to $r_1 r_2$. Once one of the end bars of P is moved close to $r_1 r_2$, the second end bar can be moved close to $r_1 r_2$ as well, and they can then both be moved close to the midpoint of $r_1 r_2$.

So we have reached a configuration where both $p_0 p_1$ and $p_2 p_3$ are at a distance at most ε from the midpoint r'_1 of $r_1 r_2$ for some appropriate $\varepsilon > 0$. Let H be the plane containing p_1, p_2 and r'_1, and project P onto H in the direction $r_1 r_2$. For any given $\delta > 0$, we can choose the value of ε so that for any bar ab intersecting H at a distance $> \delta$ from any point of the projection of P, that segment does not touch any bar of P.

Let r'_i be the intersection of $r_i r_{i+1}$ with H. If we fix the position of $r_1 r_2$, the possible positions of $r_0 r_1$ and $r_2 r_3$ intersect H in two curves (conic sections). Both these curves

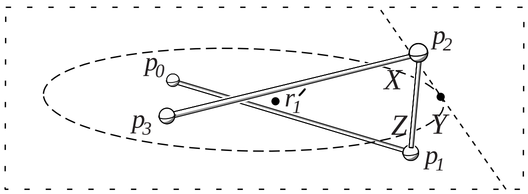

Figure 2: Possible positions for r_0' on components of the dotted ellipse in H.

are cut into pieces by the projection of P. Those pieces will be called *components* for r_0' or r_2'.

We will describe several motions of the chain P where p_1p_2 will remain in H and will be translated in some specified direction, while the support lines of p_0p_1 and p_2p_3 will slide around r_1' and remain within a distance ε of that point. We will call any such motion *feasible* if there exists a simultaneous motion of R, with r_1r_2 fixed, that introduces no crossings. This motion will not introduce crossings between P and itself, or between P and r_1r_2. Also, r_0r_1 and r_2r_3 only intersect if the rays $\overrightarrow{r_1'r_0'}$ and $\overrightarrow{r_1'r_2'}$ are equal, so we will have to preserve the radial ordering of r_0' and r_2' with respect to r_1' during the motion. The last kind of possible crossings would be between the end bars of R and the chain P. For those, we observe the possible movements of those end bars, which correspond to the *components* for r_0' or r_2'. If the component for r_0' or r_2' is unbounded (e.g. the end of a parabola), then the corresponding bar can be moved to stop intersecting H, which can only help moving P away. If the component for r_0' or r_2' is bounded but never dissapears during the entire motion, the corresponding bar can be continuously moved within that component to avoid crossings with P. So, if r_0' and r_2' are contained in unbounded components, or in bounded components that never disappear during the entire motion, then the motion is feasible. Conversely, the only way for a motion not to be feasible is when either r_0' or r_2' is contained in a bounded component that disappears. Because the curves are convex, and r_1' is inside their convex hull, the disappearance of a component during a motion of the kind described above must involve p_1p_2.

Fig. 2 denotes by X, Y and Z the three kinds of components that could disappear. Since we have only two points to place in those components, at least one of X, Y or Z contains neither r_0' nor r_2', and perhaps does not exist. If X is empty or non-existent, then we can translate p_1p_2 in the direction p_2p_1. This translation does not reduce the size of Z until p_1p_2 stops bounding Z, and Y remains unchanged by the motion, and so the motion is feasible. If Z is empty or non-existent, then translating p_1p_2 in the direction p_1p_2 produces a feasible motion for the same reasons. If Y is non-existent for at least one of the two curves, then X and Z are the same component for that curve and we fall into the previous case. Finally, suppose Y exists and is empty for both curves, and there is a non-empty X component and a non-empty Z component. Assume that X contains r_0' and Z contains r_2'. Then one of the two curves must be an ellipse; assume that it is the curve containing r_0'. We can translate p_1p_2 with r_0' along its component, away from r_1', until the Y component of r_0' disappears, connecting the X and Z components of r_0' and falling back into the previous case. □

3. INTERLOCKED CHAINS

To show that two or more chains are interlocked we establish geometric invariants, often regarding the convex hull of selected vertices or joints. We begin with some useful preliminaries. We use a bracket $[abcd]$ to denote the 4×4 orientation determinant of the homogeneous coordinates of four points a, b, c, and d. It will be positive if the ray $\overrightarrow{ab}$ is consistently oriented with ray $\overrightarrow{cd}$ according to a right-hand rule.

Since we are concerned with invariants under motion, the points in a bracket will move over time. We can make statements about the invariance of faces of convex hulls like the following two lemmas; see Figure 3 for an illustration.

Lemma 7. *Under continuous motion of a, b, c, and d, determinant $[abcd]$ is positive iff the convex hull $CH(a,b,c,d)$ is a tetrahedron with edges to a, b, and c appearing in counterclockwise (ccw) order around d.*

Proof. This is a consequence of properties of the orientation determinant. □

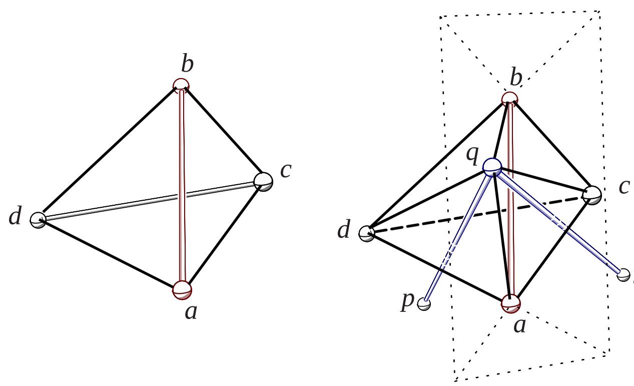

Figure 3: The configurations for Lemmas 7 and 8.

Lemma 8. *Suppose, as depicted at the right of Figure 3, that the convex hull $CH(a,b,c,d,q)$ initially has six faces $\triangle qac$, $\triangle qcb$, $\triangle qbd$, $\triangle qda$, $\triangle adc$, and $\triangle bcd$. As long as three conditions hold under motion of a, b, c, and d—that $\triangle adc$ and $\triangle bcd$ are faces of convex hull $CH(q,a,b,c,d)$, that bar pq intersects $CH(a,b,c,d)$ with $[pqab] > 0$, and that qr intersects $CH(a,b,c,d)$ with $[qrab] > 0$—the convex hull $CH(a,b,c,d,q)$ retains its face structure. In particular, ab pierces $\triangle qcd$.*

Proof. Since $\triangle adc$ and $\triangle bcd$ remain faces of the convex hull $CH(q,a,b,c,d)$, they remain faces of $CH(a,b,c,d)$, which must be a tetrahedron. By Lemma 7, $[abcd] > 0$.

We claim that q remains in the intersection of halfspaces bounded by planes through acd, bcd, abd, and acb. These planes are indicated by dotted lines at the right of Figure 3. If point q would exit this intersection by first reaching planes through acd or bcd, then a or b would no longer be a vertex of the convex hull $CH(q,a,b,c,d)$. If q first reached abd or acb, then pq or qr could no longer intersect the tetrahedron $abcd$ and maintain a positive orientation determinant with ab. (Note that reaching two or more planes simultaneously still violates the conditions.) Thus, q remains on the convex hull and keeps all its incident faces. □

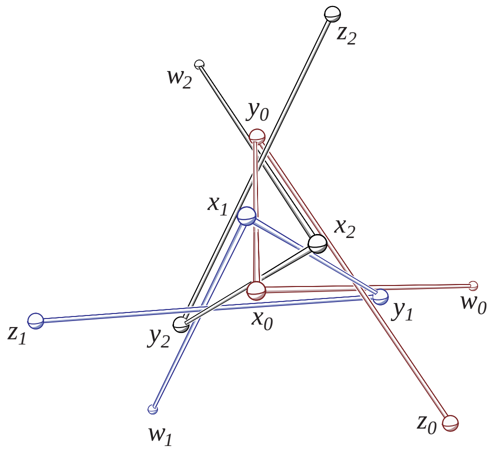

Figure 4: Three flexible 3-chains that interlock.

3.1 Three flexible 3-chains can interlock

In this section we show that the three open 3-chains of Fig. 4 interlock. We say that chain i, for $i \in \{0, 1, 2\}$ has vertices w_i, x_i, y_i, and z_i, as illustrated. We will use index arithmetic modulo 3.

To make this example, one could start with Borromean rings made of triangular chains with $w_i = z_i$, then extend the end bars of chain i above and below the surrounding chain $(i+1)$ until the end bars are at least three times longer than the middle bars. Let us assume that the middle bars have unit length.

Theorem 9. *Three flexible 3-chains can interlock.*

Proof. We can make a number of initial geometric observations, which we will show are geometric invariants of this linkage. When we say a segment *pq pierces a triangle* $\triangle abc$, it is a shorthand for saying that five brackets are positive: $[pabc]$, $[abcq]$, $[pqab]$, $[pqbc]$, and $[pqca]$—that is, points p and q are on opposite sides of the plane abc and $\triangle abc$ is oriented consistent with a right-hand rule around pq. We have the following for all $i \in \{0, 1, 2\}$:

(1) The convex hull of the joints $\mathcal{Q} = CH(\{x_j, y_j \mid 0 \leq j \leq 2\})$ is an octahedron with edges to x_{i+1}, y_{i-1}, y_{i+1} and x_{i-1} appearing counter-clockwise (ccw) around x_i and clockwise (cw) around y_i.

(2) Middle bar $x_i y_i$ pierces $\triangle x_{i-1} y_{i-1} x_{i+1}$.

(3) End bar $x_i w_i$ pierces $\triangle y_{i-1} x_{i-1} y_{i+1}$, forms positive determinants $[x_i w_i x_{i+1} y_{i+1}]$ and $[x_i w_i y_{i+1} z_{i+1}]$, and exits the hull $\mathcal{Q}$.

(4) End bar $y_i z_i$ pierces $\triangle x_{i-1} y_{i-1} y_{i+1}$ (the same triangle with the opposite orientation), forms positive determinants $[y_i z_i z_{i+1} y_{i+1}]$ and $[y_i z_i y_{i+1} x_{i+1}]$, and exits the hull $\mathcal{Q}$.

As the points and vertices move, let us consider which of these conditions could fail first. We divide them into two classes: *hull conditions*, where a joint or end point goes inside the hull $\mathcal{Q}$ or a hull edge disappears as two adjacent faces become coplanar, and *piercing conditions*, where a bar fails to pierce its triangle or one of its orientation determinants becomes zero.

We begin by showing that the first change cannot be a joint disappearing inside the convex hull. Consider vertex x_i. Segment $x_i w_i$ pierces $\triangle y_{i-1} x_{i-1} y_{i+1}$ and segment $x_i y_i$ pierces $\triangle x_{i-1} y_{i-1} x_{i+1}$. Since both enter the tetrahedron formed by the middle bars $x_{i-1} y_{i-1}$ and $x_{i+1} y_{i+1}$, we can apply Lemma 8 to the 2-chain $w_i x_i y_i$ to see that joint x_i cannot be first joint to disappear inside the convex hull. Similarly, the two segments $x_i y_i z_i$ intersect the convex hull of the two middle bars such that we can apply Lemma 8 and show that joint y_i cannot be the first inside.

If a convex hull edge disappears, then two adjacent triangles become coplanar. By the pigeonhole principle, two of the vertices of that quadrilateral are from the same chain, which implies that a middle bar $x_i y_i$ is on the convex hull. But as long as $x_i y_i$ pierces its triangle, it cannot be on the convex hull. We show below that the triangle piercing is invariant.

First, however, we argue that end points w_i and z_i never enter the hull, by establishing that the hull diameter is less than three as long as 1) and 2) hold.

Lemma 10. *If the diameter of the convex hull $\mathcal{Q}$ is ≥ 3, then either $\mathcal{Q}$ contains a joint, or a middle bar lies on the boundary of $\mathcal{Q}$.*

Proof. If the hull diameter is three or more, introduce two planes perpendicular to the diameter segment that cut it into three equal pieces. These planes cut the hull $\mathcal{Q}$ into three pieces; by the pigeonhole principle, either the first or the third piece contains (the interior of) only one middle bar $x_i y_i$. If both joints of this bar are on the convex hull, then this bar lies on the hull because the defining plane separates these joints from the remaining. □

Thus, the first failures must be piercing conditions, possibly accompanied by an edge (but not a vertex) disappearing from the hull. Without loss of generality, we consider that among the first piercing conditions to fail is one for a bar on chain 1. In preparation for finding a contradiction, we draw the projections of relevant bars from the perspectives of joints y_1 and x_1 in Figure 5, just before any piercing condition fails.

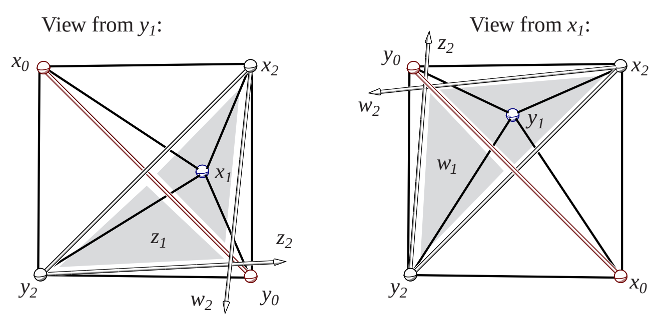

Figure 5: Views of selected bars and hull edges from y_1 and from x_1.

Consider the projection of the octahedron from y_1. By (1), we see a convex quadrilateral $y_2 y_0 x_2 x_0$ oriented ccw. By (2), point x_1 is initially inside $\triangle x_0 y_0 x_2$; since $x_2 y_2$ pierces $\triangle x_1 y_1 x_0$, we also know that x_1 is inside $\triangle x_2 y_2 y_0$. By (3), bar $w_2 x_2$ pierces $\triangle x_1 y_1 y_0$, so the projection of $w_2 x_2$ has x_1 to the left and y_0 to the right, which restricts the placement of x_1 as in the figure.

Now, suppose that the condition that x_1y_1 pierces $\triangle x_0y_0x_2$ is among the first to fail—that is, one or more of its five orientation determinants become zero. We show that each case contradicts a known property. (We do one case analysis in detail to as illustration.) We know that $[x_1x_0y_0x_2] > 0$ and $[x_0y_0x_2y_1] > 0$, since the triangle is strictly inside the convex hull and both vertices x_1 and y_1 are on the boundary of $\mathcal{Q}$. Thus, it follows from Lemma 7 that $[x_1y_1x_0y_0]$ can become zero only if bars x_1y_1 and x_0y_0 are touching. Bracket $[x_1y_1y_0x_2]$ can become zero only if the projection of x_2w_2 has moved to be disjoint from the projection of x_1y_0, meaning that the piercing condition for x_2w_2 has previously failed. Finally, $[x_1y_1x_2x_0]$ can become zero only if the condition that x_2y_2 pierces $\triangle x_1y_1x_0$ has previously failed. This establishes that the piercing condition for x_1y_1 cannot be among the first to fail.

We make a similar argument in the projection from y_1 for the piercing conditions for y_1z_1. By (4), point z_1 projects to the left of x_2y_2 and y_0x_0. Because y_2z_2 pierces $\triangle x_1y_1y_0$, the projection of y_2z_2 has y_0 to the right and x_1 to the left; the orientation determinant also says that, in projection, z_1 is to the left of y_2z_2. Thus, z_1 is restricted to the shaded region. Since y_1z_1 goes through the hull, Lemma 7 implies that the y_1z_1 will touch the bars x_0y_0, x_2y_2, or y_2z_2 if their corresponding brackets go to zero. Thus, z_1 can leave the shaded region only by touching a bar or by a previous failure of a piercing condition. Notice that the points in the shaded region satisfy all the conditions imposed upon y_1z_1 in (4).

The argument for x_1w_1 is similar and establishes that there can be no first failure of piercing conditions. This completes the proof of Theorem 9.

3.2 A 3-chain and 4-chain can interlock

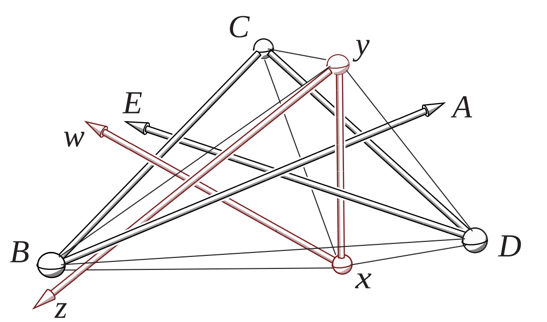

Figure 6: An example showing a locked 3-chain and 4-chain. Added lines show that the convex hull of joints is a bi-pyramid.

THEOREM 11. *Open flexible 3- and 4-chains can interlock.*

PROOF. Figure 6 depicts the core of two linked chains, $ABCDE$ and $wxyz$, where bars between joints have unit length and end bars have length greater than $BC + CD + xy = 3$. We analyze the convex hull of the flexible joints $\mathcal{Q} = CH(B, C, D, x, y)$ as the points move.

In the initial embedding of Figure 6, we make several observations that we will show are invariants. Recall that a statement that, for example, xy pierces $\triangle DCB$ is shorthand for saying that five orientation determinants are positive: $[xDCB]$, $[DCBy]$, $[xyDC]$, $[xyCB]$, and $[xyBD]$.

(1): Bar xy pierces $\triangle DCB$. Equivalently, the hull $\mathcal{Q}$ is a bi-pyramid, with edges to B, C, and D in ccw order around x and cw order around y.

(2): End bar DE pierces $\triangle Byx$ and hull face $\triangle BCx$ and makes positive orientation determinant $[DExw]$.

(3): End bar BA pierces $\triangle Cyx$ and hull face $\triangle DCy$ and makes positive determinants $[BAzy]$ and $[BAxw]$.

(4): End bar xw pierces $\triangle DCB$ and hull face $\triangle CBy$ and makes positive determinant $[wxyz]$.

(5): End bar yz pierces $\triangle BCD$ and hull face $\triangle BCx$.

Any motion that separates these chains must change the convex hull $\mathcal{Q}$ and invalidate observation (1), so some set of observations must be first to fail. We show by finding contradictions that none of these can be among the first, establishing that there is no separating motion. Unfortunately, this configuration has no symmetries to cut down on the number of cases.

To begin, we apply Lemma 8 to argue that the first event cannot include x or y vanishing inside the hull $\mathcal{Q}$. Consider x first. Since xw and xy pierce $\triangle DCB$, both bars intersect tetrahedron $CH(B, C, D, E)$. Since $[DEyx]$ and $[DExw]$ are positive, we can apply Lemma 8 to show that x cannot vanish into tetrahedron $CH(B, C, D, E)$ without some other hull change occurring. But vanishing into $CH(B, C, D, E)$ would be necessary before x could vanish into hull $\mathcal{Q}$. Similarly, yx and yz pierce $\triangle BCD$ and straddle BA, so Lemma 8 implies that the point y cannot vanish into the tetrahedron $CH(A, B, C, D)$ unless $\mathcal{Q}$ has already changed.

Next, we show that (1) cannot be among the first conditions to fail; that xy must remain inside the hull. Since we know that x and y remain on opposites sides of the plane BCD, we can most easily to argue about orientation determinants in 3D by considering projections from the perspectives of one of the joints, as illustrated in Figure 7. Consider

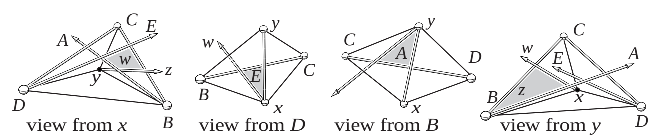

Figure 7: Projections of the linkage of Figure 6 from x, D, B, and y.

the view from x, where we see y inside a ccw-oriented triangle $\triangle BCD$. By condition (2), DE pierces $\triangle BCx$, and $[DEyx]$ is positive (from DE piercing $\triangle Byx$); these further restrict y to lie in a triangle formed by the projections of bars BC, CD and DE. By Lemma 7, the projection of y cannot reach the projections of BC or CD without causing bars to intersect. Nor can it reach reach BD without causing bars xy and DE to intersect unless there has been a previous failure of DE to pierce $\triangle BCx$, violating condition (2). Thus, condition (1) cannot be among the first condition to fail.

As long as the hull keeps its structure, we can make an argument like that of Lemma 10 to show that the diameter of the hull is at most three, which implies that end vertices never enter the hull. For end bar piercing conditions, therefore, we can continue to consider projections from joints, without worrying that a joint or end vertex will disappear inside the hull.

To see that condition (2) cannot be among the first to fail, consider the view from D, where we see a convex quadrilateral $xCyB$ whose diagonals are bars that restrict the point that is the projection of DE. By (4) and (1), bars xw and xy pierce $\triangle DCB$, so there is a triangle formed by projections of bars xw, xy, and BC that contains the projection of E. For this point to leave the projection of $\triangle Byx$ or $\triangle BCx$ or change the sign of $[DExw]$, bar DE would intersect bars xw, xy, or BC inside $\mathcal{Q}$, or the condition of (4) that xw pierces $\triangle DCB$ would have previously failed.

For condition (3), we have a similar case in the view from B. If the projection of A were to leave the projection of $\triangle Cyx$ or $\triangle DCy$, bar BA would intersect bar xy, yz, or CD, or there would have been a previous failure of condition (5), that yz pierces $\triangle BCD$.

For the piercing conditions of (4), it is sufficient to establish that xw always pierces $\triangle CBy$, because as long as it is satisfied and (1) xy pierces $\triangle DCB$, we automatically have xw piercing $\triangle DCB$. We must also establish that $[wxyz] > 0$ as points move. Consider once again the view from x. Bar xw projects to a point in a region bounded by the projections of CB, zy, BA, and DE as long as (4) bar xw pierces $\triangle CBy$, (5) bar yz pierces $\triangle BCx$ and satisfies $[wxyz] > 0$, (3) bar BA pierces $\triangle Cyx$, and (2) $[DExw] > 0$. Since xw cannot intersect bars CB, zy, BA, or DE, for the projection of w to leave $\triangle CBy$ or cause $[wxyz]$ to become negative, a piercing condition from (5) or (3) must have previously failed. Thus condition (4) cannot be among the first to fail.

For (5), consider the view from y. As long as xy pierces $\triangle DCB$, bar yz piercing $\triangle BCx$ is the more restrictive condition. Bar yz projects to a point in a triangle bounded by projections of AB, CB, and xw, since (3) AB pierces $\triangle Cyx$ and (4) xw pierces $\triangle CBy$. (In this case, we cannot use the condition $[DExw] > 0$, since the projection of point E could lie inside $\triangle CBx$.) Since yz cannot intersect bars AB, BC, or xw, the only way to leave $\triangle BCx$ would be after a previous failure of piercing conditions from (3) or (4).

Since no event can occur among the first events, we know that any motion will preserve the triangles of the convex hull $\mathcal{Q}$, and that the chains remain interlocked. □

3.3 2-rigid + 3-rigid can interlock

The remaining subsections investigate interlocking configurations with restricted motion.

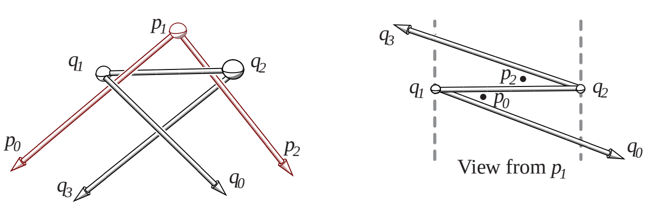

Figure 8: A rigid 2-chain and a rigid 3-chain can interlock.

THEOREM 12. *A rigid 2-chain can interlock with a rigid 3-chain.*

PROOF. The starting configuration is as shown in Fig. 8. For the two chains $P = (p_0, p_1, p_2)$, and $Q = (q_0, q_1, q_2, q_3)$, we assume that point $q_1 = (0, 0, 0)$, point $q_2 = (1, 0, 0)$, bar q_0q_1 goes through the point $q_0' = (1, -1, -1)$, bar q_2q_3 goes through the point $q_3' = (0, 1, -1)$, and all end bars have length L. The vertex angle at p_1 is $\pi/2 < \beta < \pi$. Draw a central projection of the configuration onto the xy plane from viewpoint p_1, as in Figure 8.

In the starting configuration, both bars of P intersect $\mathcal{T} = CH(q_0', q_1, q_2, q_3')$, and during any separating motion, those bars must cease intersecting $\mathcal{T}$. The diameter of $\mathcal{T}$ is less than 3, so if $L > 3/\tan\beta$, we know that if p_0 or p_2 enter $\mathcal{T}$, then one of the end bars of P will have already left $\mathcal{T}$. So during any separating motion, one of p_0 or p_2 will have to cross one of the dotted lines in the projection shown in Figure 8. Note that before the motion starts, the dot product of the planar vectors in the projection $p_0p_2 \cdot q_1q_2 > 0$, and as soon as one of p_0 or p_2 intersects one of the dotted lines in the projection, $p_0p_2 \cdot q_1q_2 < 0$. Since this is a continuous motion, at some instant we have $p_0p_2 \cdot q_1q_2 = 0$; that is, the plane containing 2-chain P is perpendicular to q_1q_2.

Consider the intersections of Q with the plane containing P at that instant. The intersection with q_1q_2 is at $(y, z) = (0, 0)$, the intersection with q_0q_1 lies on the segment joining $(0, 0)$ to $(-1, -1)$, and the intersection with q_2q_3 lies on the segment joining $(0, 0)$ to $(1, -1)$. Thus, the support line of p_0p_1 would have to be below $(-1, -1)$ and above $(0, 0)$ and the support line of p_1p_2 would have to be above $(0, 0)$ and below $(1, -1)$. But this would imply that $\beta < \pi$ which contradicts the fact that P is rigid. □

3.4 2-rigid + 4-flexible can interlock

Consider the rigid 2-chain $P = (p_0, p_1, p_2)$ and flexible 4-chain $Q = (q_0, \ldots, q_4)$ shown in Fig. 9. The lengths of the internal edges $k_1 = q_1q_2$, and $k_2 = q_2q_3$ are unity, and the length of all end bars is set to some large value L to be determined later. Let $\mathcal{T}$ be the tetrahedron with vertices $\{p_1, q_1, q_2, q_3\}$. We show:

LEMMA 13. *Starting from the configuration portrayed at the left of Fig. 9, consider any motion where none of the vertices p_0, p_2, q_0 or q_4 ever enter the tetrahedron $\mathcal{T}$. Then at all times, the edges p_0p_1 and p_1p_2 both intersect triangle $q_1q_2q_3$.*

PROOF. Along with the conclusion stated in the lemma, we will show that a few other conditions remain true at all times during the motion:

$$\begin{array}{ll} [q_0q_1q_2q_3] < 0, & [q_1q_2q_3q_4] > 0 \\ [p_0p_1q_0q_1] < 0, & [p_0p_1q_iq_{i+1}] > 0 \text{ for } i = 1, 2, 3 \\ [p_1p_2q_3q_4] > 0, & [p_2p_1q_iq_{i+1}] > 0 \text{ for } i = 0, 1, 2 \end{array}$$

and the edges p_0p_1 and p_1p_2 intersect triangle $\triangle q_1q_2q_3$, q_0q_1 intersects $\triangle p_1q_2q_3$ and q_3q_4 intersects $\triangle p_1q_1q_2$. We prove this by showing that none of these conditions can be the first one to become false. Note that these conditions also imply that p_1 remains above the plane containing $\triangle q_1q_2q_3$.

First consider all determinants involving p_0p_1 or p_1p_2. For this, we project the configuration from the viewpoint p_1 onto the $\triangle q_1q_2q_3$ as in the middle of Fig. 9. Let q_0' be the intersection of the edge q_0q_1 and the triangle $\triangle p_1q_2q_3$, q_0' projects to the intersection point of the projections of the edges q_0q_1 and q_2q_3. Likewise, let q_4' be the intersection of the edge q_3q_4 and the triangle $\triangle p_1q_1q_2$; point q_4' projects to the intersection point of the projections of the edges q_3q_4 and q_1q_2. Also, let r be the projection of the intersection between the projections of the edges q_0q_1 and q_3q_4; point r is the projection of points on those two edges that lie inside $\mathcal{T}$.

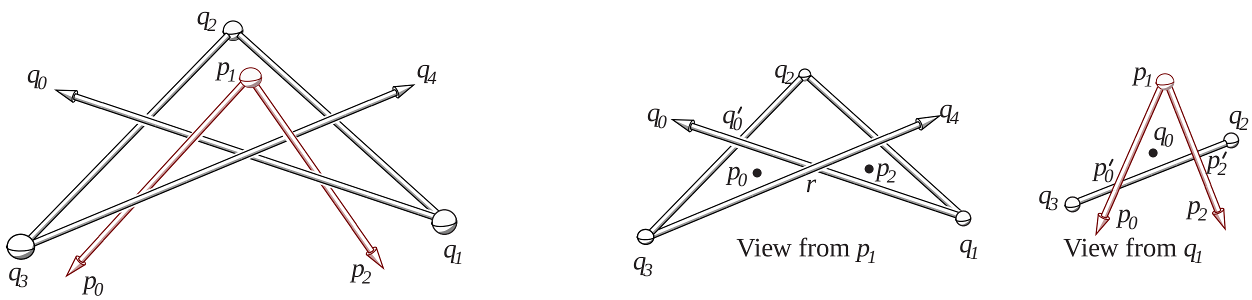

Figure 9: A rigid 2-chain P and a flexible 4-chain Q, with views from vertices p_1 and q_1.

In the projection, p_0 becomes a point lying inside the triangle $\triangle rq_0'q_3$. The three edges of this triangle are the projection of portions of edges completely contained in $\mathcal{T}$, and p_0 is not contained in $\mathcal{T}$, so none of the determinants involving p_0p_1 can change sign as the first violated condition without involving an edge crossing. The same argument can be made about edge p_1p_2 and triangle $\triangle rq_4'q_1$. The same projection also shows that the edges p_0p_1 and p_1p_2 will not stop intersecting triangle $\triangle q_1q_2q_3$ before some determinant involving one of these two edges changes sign.

For the events involving q_0q_1, we project the configuration from the viewpoint q_1 onto the $\triangle p_1q_2q_3$ as in the right of Fig. 9. Let p_0' be the intersection of p_0p_1 and $\triangle q_1q_2q_3$, p_0' projects to the intersection point of the projections of the edges p_0p_1 and q_2q_3. Let p_2' be the intersection of p_1p_2 and $\triangle q_1q_2q_3$, p_2' projects to the intersection point of the projections of the edges p_1p_2 and q_2q_3. In the projection, q_0 becomes a point lying inside the triangle $\triangle p_0'p_1p_2'$. The three edges of this triangle are the projection of portions of edges completely contained in $\mathcal{T}$, and q_0 is not contained in $\mathcal{T}$, so none of the determinants involving q_0q_1 can change sign as the first violated condition without involving an edge crossing. The same projection also shows that q_0q_1 will not stop intersecting triangle $\triangle p_1q_2q_3$ before some determinant involving q_0q_1 changes sign. The events involving q_3q_4 can be treated in the same manner, and so none of the events can occur first. □

Theorem 14. *Given any angle $0 < \beta < \pi$, there is an interlocked configuration of a 2-chain with a 4-chain, if the vertex angle of the 2-chain is restricted to stay $\geq \beta$ during the entire motion.*

Proof. Consider the configuration shown at the left of Fig. 9. We show that the length L of all 4 end bars can be made large enough so that the configuration is interlocked. By the previous lemma, in order to unlock P and Q, point p_0 or p_2 must enter tetrahedron $\mathcal{T}$ through $\triangle q_1q_2q_3$. At the time one of these endpoints, say p_0, enters $\triangle q_1q_2q_3$, p_1p_2 still intersects $\triangle q_1q_2q_3$. But the closest point to p_0 on p_1p_2 is at distance $L \sin\beta$. Since the diameter of the triangle $\triangle q_1q_2q_3$ is less than 2, the configuration will be locked if $L \sin\beta > 2$, or $L > 2/\sin\beta$. □

3.5 2-flexible + 5-rigid can interlock

Theorem 15. *A flexible* 2*-chain can interlock with a rigid* 5*-chain.*

Proof. We can build this configuration with the coordinates of Figure 10 and check that initially it has positive orientation determinants $[p_1p_2q_iq_{i+1}]$, for $i \in \{0,1,2\}$, and $[p_0p_1q_iq_{i+1}]$, for $i \in \{2,3,4\}$. The four planes $q_iq_{i+1}q_{i+2}$ for $i \in \{0,1,2,3\}$ define a tetrahedron τ, shown dotted in Figure 10, that contains p_1. We can calculate the coordinates s and t, as in the figure, so tetrahedron $\tau = CH(q_2,q_3,s,t)$.

In fact, p_1 cannot leave τ without causing bars of the two chains to intersect. Consider the view from p_1. Ends p_0 and p_2 project to points that are contained in triangles that are projections of bars of Q. These projected triangles are invariant as long as p_1 is in τ: Because the planes $q_0q_2q_3$ and $q_5q_3q_2$ completely contain τ, the end bars of Q project onto q_2q_3 until two edges of a projected triangle becomes collinear, which occurs only if p_1 reaches a face of τ. But this would also force an intersection in the projection between an end bar of P and a bar of Q. Since the length of the end bars of P is > 9, and the greatest distance of τ from a point of the projected triangle is $|sq_1| = 6$, the bars do intersect, as promised. □

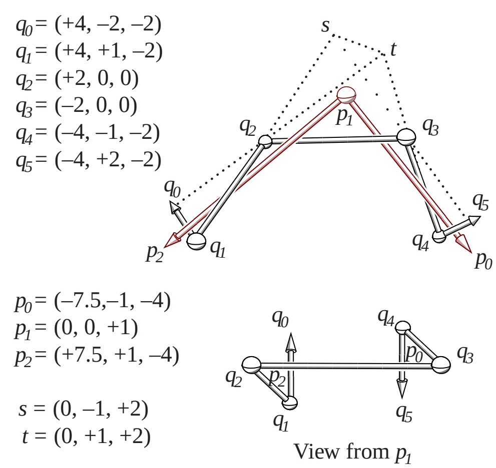

Figure 10: A flexible 2-chain and a rigid 5-chain can interlock.

3.6 3-rigid + 3-flexible can interlock

As shown in Section 2.1, two flexible 3-chains cannot interlock. To obtain a locked configuration for two 3-chains, we could restrict the motion of the chains in several ways. To make these ways precise, consider a 3-chain with vertices p_0, p_1, p_2, and p_3, and define

- the *vertex angle at* p_i, for $i = 1, 2$, which is the angle $\angle p_{i-1}p_ip_{i+1}$, and
- the *dihedral angle* of the 3-chain, which is the angle between the orthogonal projections of p_0p_1 and p_2p_3 onto a plane perpendicular to p_1p_2.

In a flexible chain, these angles are completely unrestricted. For a revolute chain, the vertex angles cannot change during the motion. We will prove that two 3-chains can be locked if:

- The sum of the two vertex angles for each chain is bounded from above by some angle $\alpha < \pi$, or
- Each of the three angles of one of the chains is bounded from below by some angle $\beta > 0$, the other chain being completely flexible.

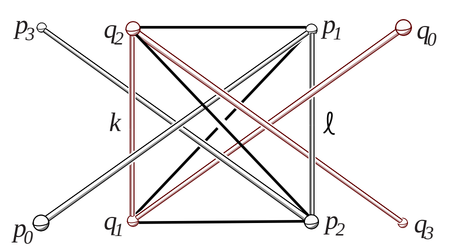

Figure 11: Two 3-chains that interlock if the joints are restricted.

Consider the 3-chains $P = (p_0, \ldots, p_3)$ and $Q = (q_0, \ldots, q_3)$ shown in Fig. 11. The lengths of middle edges $\ell = p_1p_2$ and $k = q_1q_2$ are unity, and the length of all end bars is set to some large value L to be determined later. Let $\mathcal{T}$ be the tetrahedron with vertices $\{p_1, p_2, q_1, q_2\}$. We first show:

Lemma 16. *Starting from the configuration of Fig. 11, consider any motion where none of the vertices p_0, p_3, q_0 or q_3 ever enter the tetrahedron $\mathcal{T}$, then at all times,*

$$[p_ip_{i+1}q_jq_{j+1}] \begin{cases} < 0 \text{ for } i = j = 1 \\ > 0 \text{ otherwise,} \end{cases} \quad (1)$$

and the end bar starting at each vertex of $\mathcal{T}$ intersects the opposite facet of the tetrahedron.

Proof. It can be verified that expression (1) is true at the starting configuration. Consider the first occurrence of an event that might cause (1) to become false. To consider $[p_0p_1q_jq_{j+1}]$ for $j = 0, 1, 2$, we project the inside of $\mathcal{T}$ from vertex p_1. This is illustrated in Figure 12.

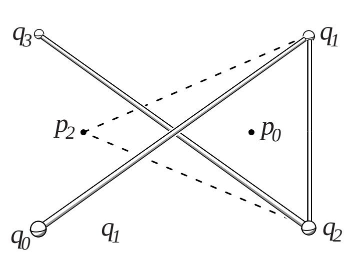

Figure 12: View from p_1

Point p_1 sees the triangle $q_1q_2p_2$ containing p_0. The segment q_0q_1 intersects $\triangle p_1p_2q_2$ and thus intersects p_2q_2 in the projection, and the segment q_2q_3 intersects $\triangle p_1p_2q_1$ and so intersects p_2q_1 in the projection. Because p_0 is actually the projection of p_0p_1, the possible projections of p_0p_1 are bounded by the segments q_0q_1, q_1q_2, and q_2q_3. All the other cases are symmetric to this one except $[p_1p_2q_1q_2]$. But this corresponds to the segments ℓ and k becoming coplanar and $\mathcal{T}$ becoming empty. But this cannot happen before one of the other events. □

Theorem 17. *Given any angle $0 < \beta < \pi$, there is an interlocked configuration of two 3-chains where the dihedral angle and both vertex angles of the first chain are $\geq \beta$ during any motion and the other chain is unrestricted.*

Proof. By Lemma 16, the dihedral angle of P is at most the angle θ between $\triangle p_1p_2q_1$ and $\triangle p_1p_2q_2$ (and thus $\theta \geq \beta$) as long as p_0, p_3, q_0 and q_3 stay out of $\mathcal{T}$. The restriction on the vertex angles of P also imply that one of the angles $\angle p_1p_2q_1$ and $\angle p_1p_2q_2$ is at least β, and the same for the angles $\angle p_2p_1q_1$ and $\angle p_2p_1q_2$. Since ℓ and k are both of length 1, then if the longest distance between any two points in $\mathcal{T}$ is D, then p_1q_1, p_1q_2, p_2q_1, and p_2q_2 are all of length $\geq D - 2$. Along with the restrictions on the angles of P, this implies that $(D - 2)\sin\beta \leq 1$ as long as p_0, p_3, q_0 and q_3 stay out of $\mathcal{T}$. Thus if we set the length L of the end bars larger than $2 + 1/\sin\beta$, p_0, p_3, q_0 and q_3 will never enter $\mathcal{T}$, and the configuration is locked. □

Corollary 18. *A rigid 3-chain and a flexible 3-chain can interlock.*

3.7 3-revolute + 3-revolute can interlock

In this subsection we consider 3-chains of Figure 11 as revolute chains, and consider the cones obtained by rotating each of the end bars around the middle edge. We will need a new lemma:

Lemma 19. *In any motion starting from the configuration of Fig. 11, the four cones defined by the chains P and Q have a non-empty intersection as long as none of the vertices p_0, p_3, q_0, or q_3 enter the tetrahedron $\mathcal{T}$.*

Proof. Using Lemma 16, we claim that the end bars of one of the chains have to intersect both cones of the other chain. To see this, observe that if, say, bar p_0p_1 does not intersect the cone at q_1, then $[p_0p_1q_0q_1]$ and $[p_0p_1q_1q_2]$ have opposite signs (because q_0q_1 is inside the cone), which contradicts lemma 16. Pick a point $\hat{q}$ at the intersection of the boundary of the two cones of Q, such that $\overrightarrow{\hat{q}q_1}$ and $\overrightarrow{q_2\hat{q}}$ have a positive orientation with p_0p_1 and p_2p_3. This implies that bars p_0p_1 and p_1p_2 both intersect the triangle $q_1q_2\hat{q}$. Construct $\hat{p}$ the same way, and notice that the triangles $p_1p_2\hat{p}$ and $q_1q_2\hat{q}$ intersect. Since the triangles are subsets of the cone intersections of their chains, this completes the proof. □

Theorem 20. *Given any angle $0 < \alpha < \pi$, there is an interlocked configuration of two 3-chains where the sum of the two vertex angles of each chain stays $\leq \alpha$ during any motion (and the dihedral angles are unrestricted).*

Proof. Let R_i, for $i = 1, 2$, be the union, over all possible pairs of vertex angles with sum $\leq \alpha$, of the intersections of the two cones of C_i. Note that R_i is contained in a sphere of radius $(\cot((\pi - \alpha)/2) + 1)/2$ centered at the midpoint of its middle bar. By Lemma 19, we know that R_1 and R_2 intersect, and so do the spheres that contain them, as long as the conditions of lemma 16 are satisfied. So if we set the length L of the end bars larger than $\cot((\pi - \alpha)/2) + 2$, then vertices p_0, p_3, q_0 and q_3 will never enter $\mathcal{T}$, and the configuration is interlocked. □

COROLLARY 21. *Two revolute 3-chains can interlock.*

4. CONCLUSION

We have settled the majority of the problems for small interlocked chains. Two problems that would complete Table 1 remain open, as well as other questions that we find interesting:

1. What is the smallest k for which a flexible k-chain can interlock with a flexible 2-chain? We believe that $6 \leq k \leq 11$.
2. What is the smallest k for which a revolute k-chain can interlock with a flexible 2-chain? Does cutting one-third of the vertices of a flexible chain suffice to separate the pieces? Corollary 3 says one-half suffices, but our results do not immediately lead to a better bound.
3. What are the interlocking configurations for sets of three or more chains with restricted motions? For example, we conjecture that a revolute 3-chain and two rigid 2-chains can interlock.
4. What is the complexity of deciding whether given chains are interlocked?

Acknowledgments

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