Ghost Chimneys

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Abstract

A planar point set S is an (i, t) *set of ghost chimneys* if there exist lines $H_0, H_1, \ldots, H_{t-1}$ such that the orthogonal projection of S onto H_j consists of exactly $i + j$ distinct points. We give upper and lower bounds on the maximum value of t in an (i, t) set of ghost chimneys, showing that it is linear in i.

1 Introduction

Once upon a time in Japan, there was a power plant with four chimneys called "ghost chimneys" (obake entotsu, or お化け煙突); see Figure 1. Although these chimneys were dismantled in 1964, they are still famous in Japan, with toys, books, manga, and movies referencing them (Figure 2). They are considered a kind of symbol of industrialized Japan in the old, good age of the Showa era [Ada09].

One of the reasons why they are famous and are called "ghost chimneys" is that they could be seen as two chimneys, three chimneys, or four chimneys depending on the point of view. This phenomenon itself was an accident, but it raises several natural questions. What interval of integers can be realized by such chimneys? How many chimneys do we need to realize the interval? How can we arrange the chimneys to realize the interval?

More precisely, we consider the following problem: given an integer i, what is the maximum value $t(i)$ such that there exists a set of points $S \subset \mathbb{R}^2$ and a set $H_0, H_1, \ldots, H_{t(i)-1}$ of lines where, for each $j \in \{0, 1, \ldots, t(i) - 1\}$, the orthogonal projection of S onto H_j consists of exactly $i + j$ distinct points? We prove the following result:

Theorem 1 *For any integer $i \geq 1$, $2i \leq t(i) \leq 123.33i$.*

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Figure 1: The ghost chimneys in Japan, part of the Senju Thermal Power Station (1926–1964) maintained by the Tokyo Electric Power Company. [Used with permission from Adachi City.]

Figure 2: Hello Ghosty

In addition to Theorem 1, we show that $t(1) = 2$, $t(2) = 5$, $t(3) = 9$, and $t(4) \geq 12$. These results show that neither the lower bound nor the upper bound of Theorem 1 is tight for all values of i. Theorem 1 is an immediate consequence of Lemma 1 and Lemma 4, which we prove in the next two sections, respectively.

These ghost-chimney problems relate more generally to understanding what orthogonal projections a 2D or 3D shape can have. In 2D, some closely related problems have been considered in [Ski90, MPv08]. Past explorations into structures in 3D, known variously as 3D ambigrams, trip-lets, and shadow sculptures, have focused on precise, usually connected projections [KvWW09, MP09]. Our work was originally motivated by considering what happens with disconnected projections of unspecified relative position.

2 The Lower Bound

Lemma 1 *For each integer $i \geq 1$, there exists a set $S = S(i)$ of $3i-1$ points and a set $H_0, H_1, \ldots, H_{2i-1}$ of lines such that, for each $j \in \{0, 1, \ldots, 2i-1\}$, the orthogonal projection of S onto H_j has exactly $i+j$ distinct values.*

Proof. The point set S consists of the points of an $i \times 3$ grid with the bottom-right corner removed; see Figure 3. For even j, H_j is a line of slope $j/2$. For odd j, H_j is a line of slope $-(j+1)/2$. □

3 The Upper Bound

Our upper-bound proof is closely related to Székely's proof of the Szemérdi–Trotter Theorem [Szé97]. We make use of the following version of the Crossing Lemma, which was proved by Pach, Radoicić, Tardos, and Tóth [PRTT04]:

Lemma 2 (Crossing Lemma) *Let $\beta = 103/6$, $\gamma = 1024/31827$, and let G be a graph with no self loops, no parallel edges, v vertices, and $e > \beta v$ edges. Then*

$$\mathrm{cr}(G) \geq \gamma \cdot \frac{e^3}{v^2} \ .$$

Lemma 3 *Let $t = \alpha i$, let S be a set of r points, and let $H_0, H_1, \ldots, H_{t-1}$ be a set of lines such that the orthogonal projection of S onto H_j gives exactly $i+j$ distinct values. Then, $t \leq 34$ or $r \leq \max\{4, 2/\alpha + 2 + \alpha/2\} i/\gamma$.*

Proof. Each projection direction H_j defines a set L_j of $i+j$ parallel lines, each of which contains at least one point of S. Let G be the geometric graph that contains the points in S plus t additional points $p_0, p_1, \ldots, p_{t-1}$. Two vertices in S are connected by an edge in G if and only if they occur consecutively on some line in $\bigcup_{j=0}^{t-1} L_j$. Additionally, each vertex p_j is connected to each of the $i+j$ lexically largest points on each of the lines in L_j. See Figure 4.

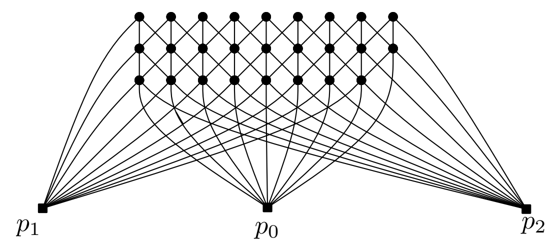

Figure 4: The graph G for a set of points with $i = 9$ and $t = 3$.

The graph G has $t + r$ vertices and tr edges. Observe that we have a drawing of G so that the only crossings between edges occur where lines in L intersect each other. The total number of intersecting pairs of lines in L is

$$\begin{aligned} X &= \sum_{j=1}^{t-1}(i+j) \cdot \sum_{k=0}^{j-1}(i+k) \\ &\leq \sum_{j=1}^{t-1}(i+j)(ij + j^2/2) \\ &\leq \sum_{j=1}^{t-1}(i^2 j + 3ij^2/2 + j^3/2) \\ &\leq i^2 t^2/2 + it^3/2 + t^4/8 \ . \end{aligned}$$

Applying Lemma 2, we learn that either

$$tr \leq \beta(t+r) \ , \tag{1}$$

or

$$X \geq \mathrm{cr}(G) \geq \gamma \frac{(tr)^3}{(t+r)^2} \ . \tag{2}$$

In the former case, we rewrite (1) to obtain

$$t \leq \beta(t/r + 1) \leq 2\beta \leq 34 + 1/3 \ ,$$

so $t \leq 34$ (since t is an integer).

In the latter case, we expand (2) to obtain

$$i^2 t^2/2 + it^3/2 + t^4/8 \geq \gamma \frac{(tr)^3}{(t+r)^2}.$$

Substituting $t = \alpha i$ gives

$$i\left(\frac{1}{2\alpha} + \frac{1}{2} + \frac{\alpha}{8}\right) \geq \gamma \frac{r^3}{(t+r)^2} \geq \gamma r/4 \ ,$$

where the second inequality follows from the fact that $t \leq i + t - 1 \leq r$. Rewriting to isolate r finally gives

$$r \leq \left(\frac{2}{\alpha} + 2 + \frac{\alpha}{2}\right) i/\gamma \ .$$

We finish the proof by observing that, for $\alpha > 2$, the inequality $r \leq 4i/\gamma$ obtained by setting $\alpha = 2$ is stronger and still applies. □

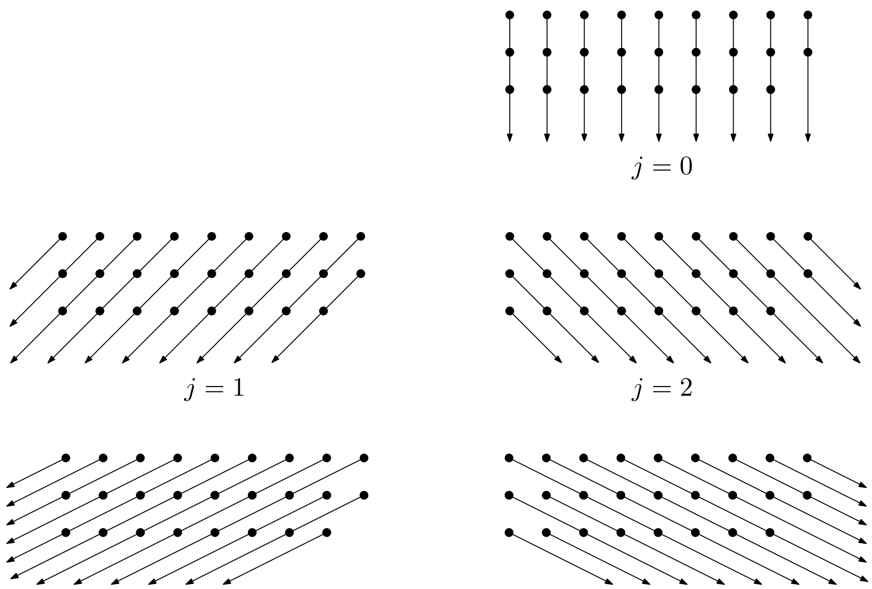

Figure 3: The set $S(i)$ for $i = 9$ and the projection directions that yield $i, i+1, \ldots, i+4$ distinct points.

Lemma 4 *For all integers $i \geq 1$, $t(i) \leq 123.33i$.*

Proof. Observe that $i+t-1 \leq r$. Therefore, for $i \geq 18$, the lemma follows by applying Lemma 3 with $\alpha = 2$. For $i \in \{1, 2, \ldots, 17\}$, the lemma follows by setting $\alpha = 35/i$. □

4 Small Values of i

In this section we give some tighter bounds on $t(i)$ for $i \in \{1, 2, 3, 4\}$.

Lemma 5 $t(1) = 2$, *and* $t(2) = 5$.

Proof. Point sets achieving these bounds are the 1×2 and the 2×3 grid, respectively; see Figure 5. That these point sets are optimal follows from the fact that the existence of H_0 and H_1 implies that the points of S lie on the intersection of i parallel lines with another set of $i+1$ parallel lines. Thus, $|S| \leq i(i+1)$, so $t(i) \leq |S| - i + 1 \leq i^2 + 1$. □

Notice that the proof of Lemma 5 implies that, for any i, $t(i) \leq i^2 + 1$. The following lemma shows that, for $i \geq 3$, $t(i) \leq i^2$. Of course, this upper bound is tighter than Lemma 4 for $i \leq 123$.

Lemma 6 $t(3) = 9$.

Proof. The point set $S(4)$ described in the proof of Lemma 1 results in 3 distinct points when projected onto a vertical line, therefore $t(3) \geq 9$.

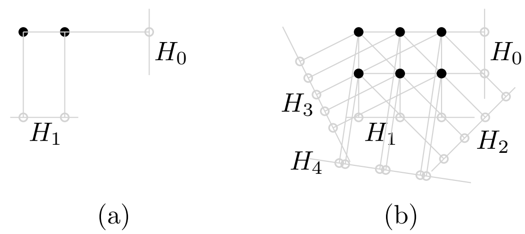

Figure 5: Point sets showing that (a) $t(1) \geq 2$ and (b) $t(2) \geq 5$.

For the upper bound, refer to Figure 6. By an affine transformation, we may assume that H_0 is vertical and H_1 is horizontal. Thus, the points of S are contained in the intersection of 3 horizontal lines with 4 vertical

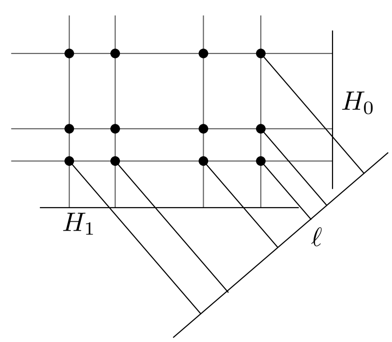

Figure 6: The proof of Lemma 6.

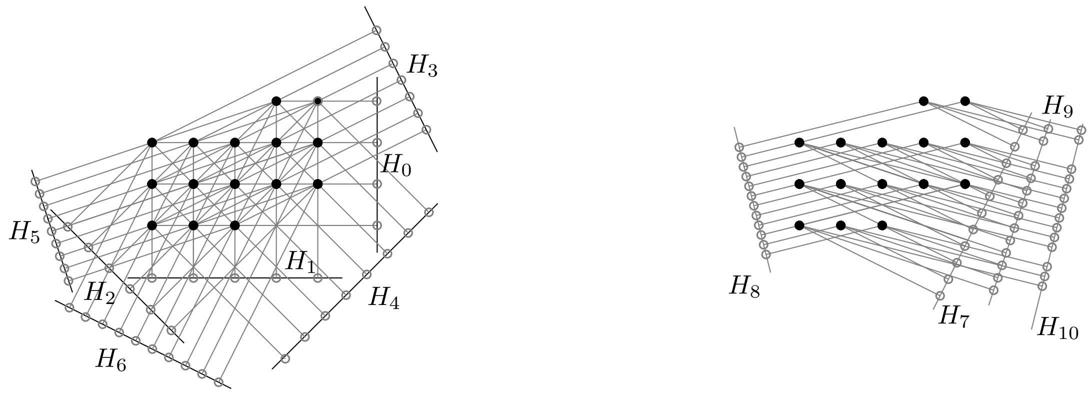

Figure 7: A $(4, 12)$ set of ghost chimneys.

lines. This establishes that $|S| \leq 12$, so $t(3) \leq 10$. To see that $|S| < 12$, assume otherwise and consider any line ℓ that is neither horizontal nor vertical. By a reflection through a horizontal line, we may assume that ℓ has positive slope, so that every point on the bottom row and right column of S has a distinct projection onto ℓ, so S projects onto at least 6 distinct points on ℓ. In particular, this implies that there is no line H_2 such that S projects onto 5 distinct points on H_2. □

Lemma 7 $12 \leq t(4) \leq 15$.

Proof. The point set and lines $H_0, H_1, \ldots, H_{10}$ that show $t(4) \geq 12$ are shown in Figure 7. (H_{11} is omitted since any sufficiently general line will do.)

To see that $t(4) \leq 15$, we argue as in the proof of the second half of Lemma 6. This establishes that $|S| \leq 20$. If $|S| \in \{19, 20\}$ then the number of distinct projections of S onto ℓ is at least 7, but this contradicts the existence of H_2. Thus, we must have $|S| \leq 18$, to $t(4) \leq 15$. □

5 Conclusions

We have given upper and lower bound on the largest possible value of t, as a function of i, in an (i, t) set of ghost chimneys. These bounds differ by only an (admittedly large) constant factor. Reducing this factor remains an open problem. For small values of i, we have shown that $t(1) = 2$, $t(2) = 5$, $t(3) = 9$, and $12 \leq t(4) \leq 18$.

Another open problem is the generalization of these results to three, or higher, dimensions. Given an integer i, what is the maximum value $t(i)$ such that there exists a set of points $S \subset \mathbb{R}^d$ and a set $H_0, H_1, \ldots, H_{t(i)-1}$ of hyperplanes where, for each $j \in \{0, 1, \ldots, t(i) - 1\}$, the orthogonal projection of S onto H_j consists of exactly $i + j$ distinct points?

Finally, the ghost chimneys of Figure 1 appear to be one chimney from some views because of the thickness of the columns. This fact suggests another problem: for what values of i and j can we place unit disks so that different views have $i, i + 1, \ldots, j$ connected components?

Acknowledgments

This work was initiated at the 25th Bellairs Winter Workshop on Computational Geometry, co-organized by Erik Demaine and Godfried Toussaint, held on February 6–12, 2010, in Holetown, Barbados. We thank the other participants of that workshop—Greg Aloupis, Brad Ballinger, Nadia Benbernou, Prosenjit Bose, Sébastien Collette, Mirela Damian, Karim Douïeb, Robin Flatland, Ferran Hurtado, John Iacono, Krishnam Raju Jampani, Anna Lubiw, Vera Sacristan, Stefan Langerman, Diane Souvaine—for providing a stimulating research environment.

References

[Ada09] Adachi City. お化け煙突 (Ghost chimneys, in Japanese). http://www.city.adachi.tokyo.jp/003/d10100040.html, 2009.

[KvWW09] J. Keiren, F. van Walderveen, and A. Wolff. Constructability of trip-lets. In *Abstracts from the 25th European Workshop on Comp. Geom.*, Brussels, Belgium, 2009.

[MP09] N. J. Mitra and M. Pauly. Shadow art. *ACM Transactions on Graphics*, 28(5), 2009. Article 157.

[MPv08] J. Matoušek, A. Přívětivý, and P. Škovroň. How many points can be reconstructed from k projections? *SIAM J. on Discrete Mathematics*, 22(4):1605–1623, 2008.

[PRTT04] J. Pach, R. Radoicić, G. Tardos, and G. Tóth. Improving the crossing lemma by finding more crossings in sparse graphs. In *Proc. of the 20th Annual Symposium on Comp. Geom.*, pages 68–75, 2004.

[Ski90] S. Skiena. Counting k-projections of a point set. *J. of Combinatorial Theory, Series A*, 55(1):153 – 160, 1990.

[Szé97] L. A. Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combinatorics, Probability and Computing*, 6:353–358, 1997.